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旋回関数の定義と旋回流解析

A Definition of swirl function and identification of swirling flow

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abstract

A method of identification of swirling flow (vortex) with definition of swirl function is presented.

In fluid motion, eigenvalue of velocity gradient tensor classifies flow characteristic, and a complex (conjugate) eigenvalue indicates that flow is swirling motion (vortex) around the point as its axis. The imaginary part represents its angular velocity of swirling, and is Galilean invariant. This quantity is defined as swirl function as a physical property.

The swirl function is a function of flow field where velocity field is defined, and the local maximum point of swirling function can be considered as its axis in finite swirling (vortical) region. Then an identification method with distribution of swirl function is developped, as SWANA2 code. This analysis is appropriate to estimate both location and intensity of swirling, and can identify vortex which the second invariant of velocity gradient tensor can not identify.

SWANA2 is verified with Burgers vortex with uniform flow, and an application in CFD (Computational Fluid Dynamics) and experiment shows that this code can identify swirling motion with concrete vortical structure of velocity even in the case that swirling motion is hidden in uniform velocity or that flow visualization (streamline) indicates swirling location different from the correct swirling region.

1 Introduction

Swirling motion or vortical flow (vortex) corresponds to many fluid problems and many engineering/design field, such as drag force behind aerofoil in aeronautical engineering, turbine blade and fluid machinery in mechanical engineering, or flow force behind structure. This vortical flow has important effect to flow characteristic and flow stability in the region to be considered. In these case, analysis for identification (checking existence) of vortical flow and for estimation of its intensity is important. In large scaled vortical flow, it is informative if the correct axis in finite or large scaled vortical region can be identified.

In spite that analyzing swirling motion is important in several engineering fields and design, the unique physical and mathematical definition of vortical flow is not established in fluid mechanics. In engineering and design field, clear definition is required to identify location and estimate intensity of the swirling motion.

In study of vortical flow, some definitions are investigated and proposed, such as eigenvalue of velocity gradient tensor,[1] the second invariant of velocity gradient tensor,[3] delta definition applying velocity gradient tensor, helicity[9][10], Hessian of pressure[5],[6] and vorticity.[8] Although several definitions are proposed, the unique definition has not been developed, then each definition can be applied in some engineering field in which the characteristic this definition is considered to be suitable.[7]

In the definition with vorticity,[8] which represents rotational component of minute element of fluid, concentrated area of vorticity is not always swirling region, such as shear flow. the second invariant of velocity gradient tensor[3] covers this pending matter with estimating the difference between the norms of vorticity tensor and of velocity gradient tensor, but this invariant does not indicate the intensity of swirling directly. Helicity[9][10] is effective in eduction of swirling motion in flow, and the angle between vortical flow and main flow. Nevertheless it is the same in a point that this does not indicate the intensity of swirling directly. The definition by Hessian of pressure[5][6] is generally difficult to apply in experiment or analysis of field data. For the application to engineering and design, the definition of vortical flow with velocity may have advantage.

Chong et. al.[1] classified of flow pattern in three dimension with eigenvalues of velocity gradient tensor using phase space of ordinary differential equation, and vortical flow is classified by complex value of eigenvalues. In the phase space, The combinations of eigenvalues and

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eigenvectors of autonomous equation indicates the characteristics of solution trajectories, and it applies to the classification of flow pattern around the point to be considered. In the case that eigenvalues include complex number, the solution (flow) trajectory swirls around the point.

The several identification methods using eigenvalues of velocity gradient tensor or phase space of autonomous equation are proposed by Sujudi et. al.,[2] Berdahl,[4] and Sawada.[11]

Sujudi et. al.[2] investigated the analysis of searching swirling motion with the eigenvalue of velocity gradient tensor, and defined the point where the velocity component is zero in swirling plane normal to swirl axis as axis point. On the other hand, generally uniform velocity may exist in swirling area and the velocity components in the axis are not zero. Then this method is difficult to extract the axis in such case.

The identification method with the ratio between complex number and uniform velocity[4] can indicate the swirling area, but it is difficult to indicate the absolute intensity of swirling, or indicate the axis of swirling motion.

Sawada[11] formulate an autonomous equation with respect to flow trajectory in a cell used for CFD (Computational Fluid Dynamics). In this formulation, the cell is supposed to be a tetrahedron and velocity components are interpolated linearly in the cell. This method applies in aeronautical engineering and turbine[10]. In the case that vortical flow is finite and covers several cells (more than one cell), it is difficult to identify the axis.

In this paper, imaginary part of complex eigenvalues of velocity gradient tensor is defined as "swirl function". This swirl function indicates the intensity of swirling (angular velocity) and this is invariant in Galilei transformation (coordinate transformation). Then swirl function can be considered as a physical property. This function has a characteristic that it has a local maximum value on the axis in Burgers vortex. Here the swirl axis is defined from the distribution of the swirl function, local maximum point in swirling region.

The identification method using this proposal enables to identify the vortical flow and its axis in spite of the size of vortical region, or existence of uniform velocity, in CFD or experiment[12].[13] This definition is effective in engineering problem with complex flow, not only in CFD analysis but also in experiment, as it requires only velocity components, not pressure. Then numerical analysis code "SWANA2" is developped in two or three dimension, which estimate velocity gradient tensor and evaluate swirl function.

Hereafter definitions of swirling motion and swirling function are described, and some application in are presented.

2 Definition of swirling motion

The definition of swirling motion is described as follows. We formulate with velocity gradient tensor,[1] and define swirl function.

When we discuss a motion that is significant physically, it must be an invariant motion in spite of coordinate transformation in inertia system (Galilei transformation). We need to define swirl motion in mathematical expression that satisfy this condition. Then it is understood that the definition of vortex with streamline does not satisfy as an integral of velocity does not have invariance.

In velocity field in three dimension given by $\upsilon_i(x)$ ($x = (x_1, x_2, x_3)$), we set a point as $\tilde{x}_i$, and consider the coordinate $\hat{x}_i$ which origin is $\tilde{x}_i$, and which moves with velocity $\upsilon_i(\tilde{x})$ ($\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$). This coordinate $\hat{x}_i$ and spatial fixed coordinate x_i (Cartesian coordinate) has relation

$$\begin{aligned}\hat{x}_i &= x_i - \tilde{x}_i \\ &= x_i - \upsilon_i(\tilde{x})t \end{aligned} \tag{1}$$

And the velocity in two coordinates has a relation

$$\hat{\upsilon}_i(\hat{x}) = \upsilon_i(x) - \tilde{\upsilon}_i \tag{2}$$

$$\tilde{\upsilon}_i = \upsilon_i(\tilde{x}) \tag{3}$$

where $\hat{\upsilon}_i$ is velocity tensor (vector) in $\hat{x}$ coordinate.

Taylor expansion of $\hat{\upsilon}_i$ derives

$$\begin{aligned}\hat{\upsilon}_i(\hat{x}) &= \hat{\upsilon}_i(0) + \frac{\partial \hat{\upsilon}_i}{\partial \hat{x}_j}\hat{x}_j + \frac{1}{2}\frac{\partial^2 \hat{\upsilon}_i}{\partial \hat{x}_j \partial \hat{x}_k}\hat{x}_j\hat{x}_k + \cdots \\ &\approxeq \hat{\upsilon}_i(0) + \frac{\partial \hat{\upsilon}_i}{\partial \hat{x}_j}\hat{x}_j \end{aligned} \tag{4}$$

neglecting higher order terms. we note

$$\hat{\upsilon}_i(0) = 0 \tag{5}$$

Substituting into eq.(4) derives

$$\hat{\upsilon}_i(\hat{x}) = \frac{\partial \hat{\upsilon}_i}{\partial \hat{x}_j}\hat{x}_j \tag{6}$$

From eq.(1) and eq.(2), velocity gradient tensor between to coordinates is equivalent, i.e.

$$\frac{\partial \hat{\upsilon}_i}{\partial \hat{x}_j} = \frac{\partial \upsilon_i}{\partial x_j} \tag{7}$$

The left hand term in eq.(6) can be expressed as :

$$\hat{\upsilon}_i(\hat{x}) = \frac{d}{dt}\hat{x}_i \tag{8}$$

Then, eq.(6) can be expressed as

$$\frac{d\hat{x}_i}{dt} = \frac{\partial v_i}{\partial x_j}\hat{x}_j \tag{9}$$

or

$$\frac{d}{dt}\hat{\boldsymbol{x}} = \boldsymbol{A}\hat{\boldsymbol{x}} \tag{10}$$

$$\boldsymbol{A} \equiv [a_{ij}] \tag{11}$$

$$a_{ij} \equiv \frac{\partial v_i}{\partial x_j} \tag{12}$$

with vector notation. This is a formula of velocity $\hat{v}_i$ around $\hat{x}$.

Eq.(10) is an autonomous equation with respect to $\hat{x}$. In autonomous equation, the solution can be expressed with respect to the corresponding eigenvalue and eigenvector, by solving the eigenequation. Then the solution can be analyzed by solution trajectory and phase space. This expresses the flow state around the point $\hat{x}_i$. We note that this flow state given by eq.(10) is invariance in Galilei transformation and then this flow characteristic has physical meaning.

The eigenequation of eq.(10) can be described as

$$\det\left|\frac{\partial v_i}{\partial x_j} - \lambda\delta_{ij}\right| = 0 \tag{13}$$

where λ is eigenvalue and δ_{ij} is Kronecker delta. In case of no compressible fluid, the continuous equation

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{14}$$

is added as a condition.

This eigenequation (13) is an equation of third order, and it has three eigenvalue. The solution trajectory of eq.(10) can be expressed with respect to eigenvalue λ_j ($j = 1, 2, 3$) and eigenvector $\boldsymbol{\xi}^{(j)} = \xi_i^{(j)}$ ($i = 1, 2, 3$) of eq.(10), i.e.

$$\hat{\boldsymbol{x}} = \sum_{j=1}^{3} c_j e^{\lambda_j t}\boldsymbol{\xi}^{(j)} \tag{15}$$

$$c_j \in \boldsymbol{R} : Const.\ (j = 1, \cdots 3)$$

For the third order equation, the solution has two case;

(i) three real numbers
(ii) one real number and two complex numbers

In the latter case, the complex number is conjugate. we set conjugate complex number as λ_1, λ_2 , and real number as λ_3

$$\lambda_1, \lambda_2 = \lambda_R \pm i\phi \tag{16}$$

$$\lambda_3 = \lambda_{axis} \tag{17}$$

(i:imaginary number)

($\phi > 0$)

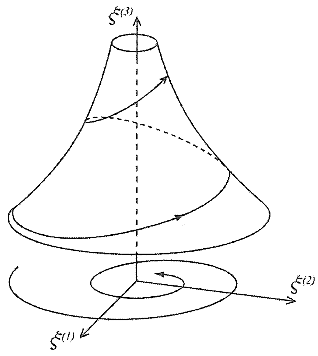

Fig.1 trajectory of swirling motion

and conjugate complex eigenvector as

$$\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)} = \boldsymbol{\xi}_{plane} \pm i\boldsymbol{\eta}_{plane} \tag{18}$$

then solution trajectory of eq.(15) is given

$$\begin{aligned}\hat{\boldsymbol{x}} &= e^{\lambda_R t}(\boldsymbol{\xi}_{plane} + i\boldsymbol{\eta}_{plane})(\cos\phi t + i\sin\phi t)\\ &+ e^{\lambda_R t}(\boldsymbol{\xi}_{plane} - i\boldsymbol{\eta}_{plane})(\cos\phi t - i\sin\phi t)\\ &+ e^{\lambda_{axis} t}\boldsymbol{\xi}_{axis}\end{aligned} \tag{19}$$

Then

$$\begin{aligned}\hat{\boldsymbol{x}} &= 2e^{\lambda_R t}(\boldsymbol{\xi}_{plane}\cos\phi t - \boldsymbol{\eta}_{plane}\sin\phi t)\\ &+ e^{\lambda_{axis} t}\boldsymbol{\xi}_{axis}\end{aligned} \tag{20}$$

Here we set $c_j = 1 (j = 1, \cdots 3)$

Eq.(20) indicates that the solution (flow) swirls in the plane defined vectors $\boldsymbol{\xi}_{plane}$ and $\boldsymbol{\eta}_{plane}$, and proceeds to the direction of vector $\boldsymbol{\xi}_{axis}$. as swirl axis. In case $\lambda_R < 0$, the flow is a swirl motion with suction (vortex) as shown in Fig. 1. This flow state given from velocity gradient tensor does not depend on existence of uniform flow.

3 Definition of swirl function

As described before, if an arbitrary point has conjugate complex number in eigenvalue of velocity gradient tensor, the flow can be considered to swirl around the point. The imaginary part of the conjugate complex eigenvalue indicates the angular velocity of swirling.

Thus we can define the imaginary part in eq.(16) as swirl function such that

$$\phi(x) = \begin{cases} \phi & \text{(case (ii))} \\ 0 & \text{(case (i)))} \end{cases} \tag{21}$$

we note that swirl function is zero where the eigenvalue of velocity gradient tensor has no imaginary part (conjugate complex number), i.e. where there has no swirling. The swirl function indicates that the flow is swirling around the point where the function has non zero value. There is no swirling motion in the area that swirling function has zero. Thus the function is a criterion of classifying swirling/non swirling flow. In addition, this represents the intensity of swirling, i.e. angular velocity of swirling. Vorticity can express the intensity of swirling, but is not appropriate for classifying the flow as it has non zero value even if flow does not swirl.

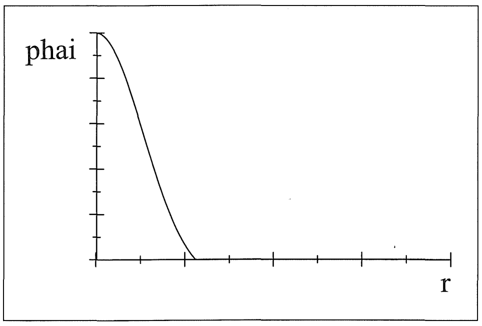

Fig.2 swirl function $\phi(r)$ in Burgers vortex

The analysis of swirling function in Burgers vortex shows that the swirling function has maximum in the centre (axis) of vortex as shown in Fig.2 (see next chapter). We define the local maximum point in an region where swirling function has non zero value as the axis of swirling motion.

4 Application

Swirl analysis is performed by calculating velocity gradient tensor and estimate eigenvalues and corresponding eigenvectors. Velocity gradient tensor is given by finite difference of velocity components in neighboring node. Then numerical analysis code "SWANA2" is developped in two or three dimension.

Application of SWANA2 in Burgers vortex and in experiment data are presented hereafter.

4.1 *Burgers vortex*

The velocity distribution of Burgers vortex is described as follows in cylindrical coordinates (r, θ, z) :

$$v_r = -\frac{\alpha}{2}r \tag{22}$$

$$v_\theta = \frac{\Gamma}{2\pi r}(1 - e^{-\frac{\alpha r^2}{4\nu}}) \tag{23}$$

$$v_z = \alpha z \tag{24}$$

α : positive constant

ν : viscosity

Γ : circulation

Fig.3 and 4 shows the velocity distribution of Burgers vortex. In the figures hereafter, velocity (or swirl function) is high in red, and low in bule. Fig. 4 shows the velocity distribution on swirling plane. In figure, velocity is high in red, and low in bule.

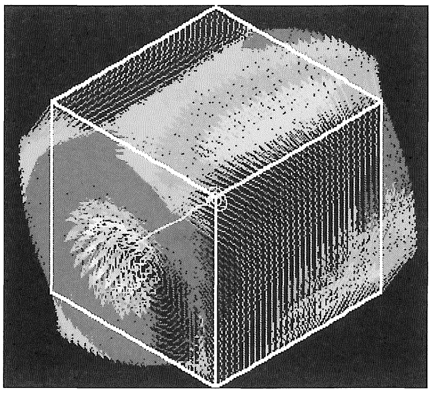

Fig.3 Burgers vortex (velocity distribution)

We compose 30 × 30 × 30 nodes and give the velocity component in Cartesian coordinates at each node in Fig.3. Fig.5 shows the contour (distribution) of swirl function on swirling plane as result of swirl analysis. It is shown that swirl function has maximum at the centre (axis).

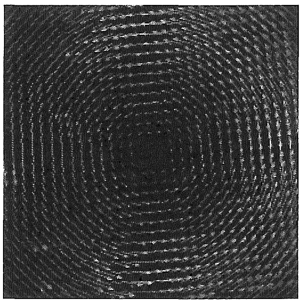

Fig.4 velocity distribution on swirling plane

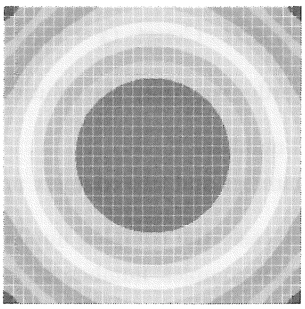

Fig.5 swirl function of Burgers vortex

If uniform velocity normal to the axis exists, the velocity distribution is given as follows:

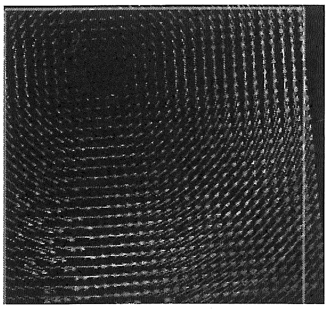

Fig.6 swirling plane with uniform velocity

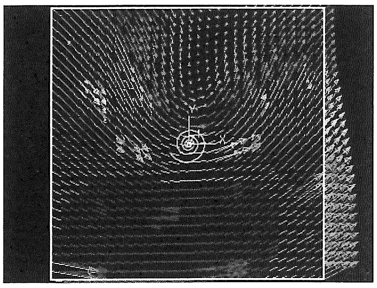

Fig.7 Burgers vortex with uniform velocity

$$v_1 = \frac{x_1}{r} - \frac{x_2}{r} v_\theta + u_1 \tag{25}$$

$$v_2 = \frac{x_2}{r} + \frac{x_1}{r} v_\theta + u_2 \tag{26}$$

$$v_3 = \alpha x_3 \tag{27}$$

$(u_{1,} u_2$: uniform velocity)

If uniform velocity normal to axis exists around vortex, the velocity distribution changes and streamline shows as if the vortex should exist in different area, as shown in Fig.6 on swirling plane. In Fig.6 the vortex seems to locate in different point, but swirl function distribution is the same as shown in Fig.5.

Also Fig.7 and Fig.8 show the velocity distribution of Burgers vortex with uniform velocity normal to the axis in three dimension. The uniform velocity in Fig.8 is larger than that in Fig.7. The velocity distribution or the stream line do not give information of existence of vortex, in spite that the vortex is still at the same location shown in Fig.3.

Fig.3, 7 and Fig.8 shows the trajectory of flow derived from eigenvectors and eigenvalues given by eq.(20), with yellow line. This trajectory is drawn near the axis that local maximum of swirl function indicates. It is shown that swirl function indicates the correct location and that the trajectory converges to the axis. The local maximum swirl function is equal to intensity of angular velocity at the axis.

The result of swirl analysis of Burgers vortex shows

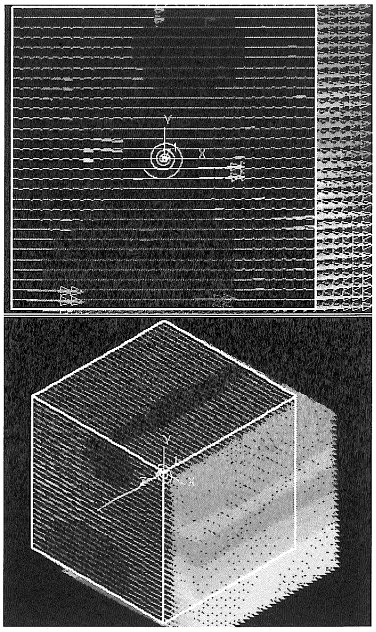

Fig.8 Burgers vortex with large uniform velocity

the possibility of misunderstanding on checking existence of swirling motion with streamline or velocity distribution, and shows that present analysis extracts (identifies) swirling motion in correct location and intensity.

4.2 *Separation vortex*

Fig.9 shows an example of separation flow and vortex in two dimension, composed of app. 3000 cells. In Fig.9, Flow pass through an substance in the lower part with 10 [m/s] and another flow is exhausted from the backside of the substance with 1 [m/s]. Then separation vortex can occur downstream. Fig.10 shows the pressure distribution. Velocity and pressure distribution does not show clearly that vortex exists, but swirl function shows a vortex downstream clearly, as shown in Fig.11.

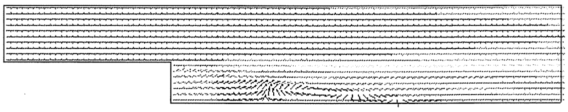

Fig.9 velocity distribution of separation vortex

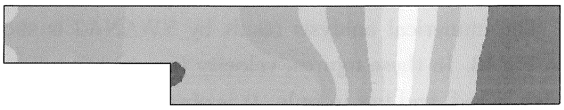

Fig.10 pressurte distribution of separation vortex

We note that pressure distribution does not always indicate in some ranges of contour. If its contour has wide

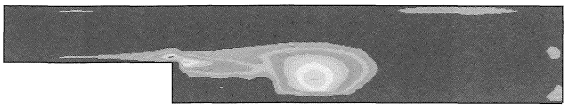

Fig.11 swirl function distribution of separation vortex

ranges, local minimum area due to vortex may be hidden.

4.3 *analysis of experimental data*

In the instrumentation of velocity field, PIV (Particle Image Velocimetry) is applied in two dimension. Fig.12 shows a velocity distribution which computer receives from PIV.

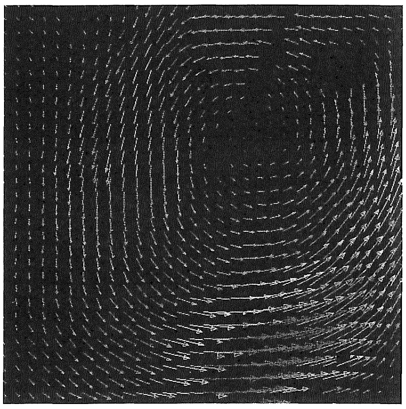

Fig.12 velocity distribution obtained by PIV

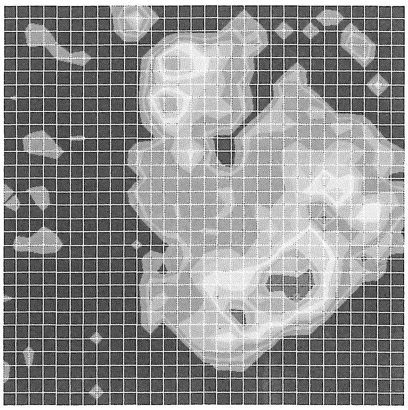

Fig.13 swirl function distribution

The numerical analysis result by SWANA2 is shown in Fig.13. In these figures, velocity (or swirl function) is high in red, and low in bule. It is observed that the swirl axis in velocity distribution in Fig.12 and the axis indicated by local maximum of swirl function differs. Then uniform velocity defined as velocity in the local maximum point is deleted in velocity distribution, and the modified velocity distribution is obtained as shown in

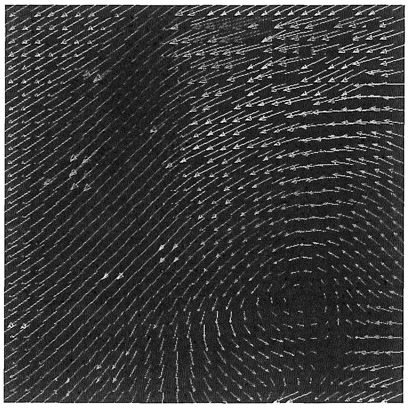

Fig.14 modified velocity distribution

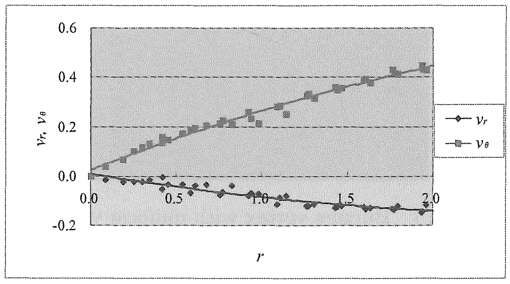

Fig.15 velocity structure indicated by swirl function (corresponding to Fig.14)

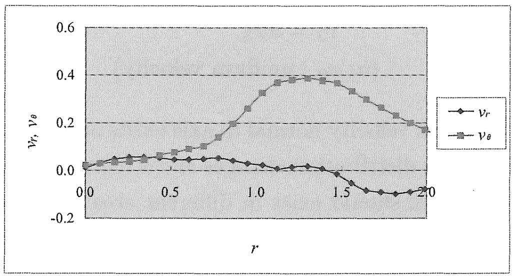

Fig.16 velocity structure indicated by velocity distribution(corresponding to Fig.12)

Fig.14. Fig. 14 shows that the local maximum point is clearly the swirl axis.

In comparison with the swirling motion observed in velocity distribution and in swirl condition, each velocity field in Fig.12 and Fig.14 is transformed into polar coordinate with origin given at the swirl axis. Fig.16 and Fig.15 shows the results respectively. The velocity structure (distribution) in Fig.15 which swirl function indicate as swirling motion (axis) show the vortical structure. The negative velocity in the radial component shows suction swirling and the circumferential components indicate that the gradient of this component is maximum at the axis, such as Burgers vortex.

On the other hand, The velocity structure (distribution), which can be visually regarded as swirling motion in Fig.12 (PIV observation), does not have characteristic of swirling motion, neither in the radial component nor circumferential component, as shown in Fig.16.

5 Discussion

Analysis of Burgers vortex and example shows that streamline or visual observation is not accurate in examination of swirling motion. But the present analysis with swirl function can identify swirling motion correctly. It is noted that swirl function is not given only in centre of swirling motion, but given in a area of swirling. We note that a point which has conjugate complex eigenvalue (non zero swirl function) is not always a centre of swirling motion. Swirl function is a function defined in velocity field, such as vorticity. Swirl function indicates an area of swirling, but and centre of swirling can be identified with the distribution of swirling function. The local maximum point in a swirling plane indicates the centre. This can be proved in case of Burgers vortex analytically, and examples of CFD results and analysis of experimental data shows this characteristic. Analysis with swirl function not only identifies swirling motion but also estimates intensity of swirling.

The present method can identify the swirling axis in spite of existence of uniform flow. Even uniform flow shows as if the swirling area is at different area or as if there is no swirling motion, this method identifies correctly. This method is effective to focus the area where we should consider to change flow state. On the other hand, the method that search zero velocity point in a swirling plane for the swirling centre is not valid where uniform velocity exists.

The application of experiment described before indicates that the examination of swirling motion by visual observation misleads in its existence and its location. It is understood that streamline does not satisfy Galilei invariance, and this is the reason of misleading. Even though flow is visualized, the verification of flow is insufficient in identification of swirling motion. The estimation of swirling motion should be examined with mathematical formulation to identify true physical behavior. Swirling motion can not be observed at all especially where large uniform velocity exists.

It is shown that the present method identifies correct location of swirling axis even if uniform velocity exists. Other identification method that search the point where velocity is zero in swirling plane[2] can not identify in such case. Another identification method which estimate the Hessian of pressure[6] is not effective in combination with experiment, as pressure distribution is difficult to be measured in experiment.

6 Conclusion

Swirl function is defined from eigenvalues of velocity gradient tensor. This property has a Galilei invariance, and indicate the angular velocity if the flow characteristic can be classified as vortical flow.

And identification method of axis of vortical flow from local maximum point of swirl function is presented. This method is applicable to estimate vortical axis in spite of size, intensity. It is also applicable in case that uniform velocity exists or vortical flow (axis) moves with non-zero velocity. It can be appropriate in analysis of experiment, as it requires only velocity data.

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