A Remark on the 2-valued Algebroid Functions

Yûji HASHIMOTO

§ 1. Introduction. Let $f = (f_0(z), f_1(z), f_2(z))$ be a transcendental system of entire functions in the finite plane $|z| < \infty$ and $X = \{(a_0, a_1, a_2) \mid a_0, a_1, a_2 \in \mathbb{C}\}$ a subset of $\mathbb{C}^3$ whose arbitrary three vectors are linearly independent.

Now, let $w(z)$ be a 2-valued transcendental algebroid function with the irreducible defining equation $F(z,w) = f_0(z)w^2 + f_1(z)w + f_2(z) = 0$, then, we have a system $f = (f_0(z), f_1(z), f_2(z))$ and a set $X = \{(w^2, w, 1) \mid w \in \mathbb{C}\} \cup \{(1,0,0)\}$.

The deficiency relation concerning f and X is given by Cartan [1] ;

$$\sum_{a} \delta(a,f) \leqq 3 + \lambda, \quad a = (a_0, a_1, a_2) \in X,$$

where λ ($=0$ or 1) is the maximum number of $\mathbb{C}$-independent linear relatins among f_0, f_1 and f_2. According to this theorem, we see that the value distribution of a system is closely concerned with its λ.

In this paper, we shall show a certain property of f for $\lambda = 1$ in the case of algebroid functions, and then give proofs of some interesting theorems obtained by Niino-Ozawa [2] and Ozawa [3] from this point of view, which is, in essential, originated in Toda [5].

We shall use the standard symbols of the Nevanlinna theory of systems and algebroid functions (see Cartan [1] and Selberg [4]).

§ 2. Now, we shall give the following theorem.

Theorem. Let $w(z)$ be as in §1. If $\lambda = 1$, then, there exists an elliptic linear fractional transformation T of period 2 and, letting w_1 and w_2 be the values $w(z)$ takes at z, we have $w_2 = Tw_1$.

Proof. Let $F(z,w) = f_0(z)\, w^2 + f_1(z)\, w + f_2(z) = 0$ be the defining equation of $w(z)$. We assume $\lambda = 1$, so that we have two cases ;

1) $f_2 = \alpha f_0 + \beta f_1$, where $\alpha \neq -\beta^2$ because $F(z,w)$ is irreducible, and

2) $f_1 = \gamma f_0$.

The case 1). Let $F(z,w_1) = f_0 w_1^2 + f_1 w_1 + f_2$ and $F(z,w_2) = f_0 w_2^2 + f_1 w_2 + f_2$, then, taking $f_2 = \alpha f_0 + \beta f_1$ into account, we have $F(z,w_1) = (w_1^2+\alpha) f_0 + (w_1+\beta) f_1$ and $F(z,w_2) = (w_2^2+\alpha) f_0 + (w_2+\beta) f_1$. Considering the condition to imply "$F(z,w_1) = 0 \Leftrightarrow F(z,w_2) = 0$", we have

$$\begin{vmatrix} w_1^2+\alpha & w_1+\beta \\ w_2^2+\alpha & w_2+\beta \end{vmatrix} = 0.$$

From this equation, we have $w_1 w_2 + \beta(w_1+w_2) - \alpha = 0$, hence, $w_2 = \dfrac{-\beta w_1 + \alpha}{w_1 + \beta} = Tw_1$. Here, by transforming the equation into $\dfrac{w_2-\omega_1}{w_2-\omega_2} = -\dfrac{w_1-\omega_1}{w_1-\omega_2}$, where ω_1 and

ω_2 are the fixed points $-\beta \pm \sqrt{\beta^2 + \alpha}$ of T, we can see that the transformation is elliptic of period 2.

The case 2). As in the case of 1), we have

$$\begin{vmatrix} w_1^2 + \gamma w_1 & 1 \\ w_2^2 + \gamma w_2 & 1 \end{vmatrix} = 0 .$$

From this equation, we have $w_1 + w_2 + \gamma = 0$, hence, $w_2 = -w_1 - \gamma = Tw_1$. This time, transforming the equation into $w_2 - \left(-\frac{\gamma}{2}\right) = -\left\{w_1 - \left(-\frac{\gamma}{2}\right)\right\}$, where $-\frac{\gamma}{2}$ is one of the fixed points of T, we can see the transformation to be elliptic of period 2.

We can also ensure the theorom by direct calculation. Q. E. D.

Concerning this theorem, we remark that, if $w(z)$ takes only one value w_0 at z (for instance, it occurs at the branch points of $w(z)$), then w_0 is a fixed point of T.

§ 3. Applying this theorem, we shall give proofs of the following interesting theorems in the theory of algebroid functions.

Theorem A (Niino-Ozawa [2] and Toda [5]). Let $w(z)$ be a 2-valued transcendental entire algebroid tunction with the deficiency relation

$$\sum_a \delta(a, w) > 2, \quad a \neq \infty,$$

then one of $\{a\}$ is a Picard exceptional value of $w(z)$.

Proof. Let $F(z, w) = w^2 + f_1(z)\, w + f_1(z) = 0$ be the defining equation of $w(z)$. We have $\sum_a \delta(a, w) > 3$ (including $a = \infty$), so that, by the theorem of Cartan in § 1, we have $\lambda = 1$ for $w(z)$. According to the theorem in § 2, we may consider the following two cases.

1). If $f_2 = \alpha \cdot 1 + \beta f_1$, then, letting w_1 and w_2 be the values $w(z)$ takes at z, we have $w_2 = Tw_1 = \frac{-\beta w_1 + \alpha}{w_1 + \beta}$. Therefore, $-\beta$, the corresponding value to ∞, is a Picard value of $w(z)$. Surely, we have

$$F(z, -\beta) = \beta^2 - f_1\beta + f_2 = \beta^2 + \alpha \neq 0 .$$

2). If $f = \gamma \cdot 1$, then, letting w_1 and w_2 be as above, we have $w_2 = Tw_1 = -w_1 - \gamma$. In this case, ∞ being a fixed point of T, there is no corresponding point. However, since ∞ is a Picard value of $w(z)$, $w(z)$ takes another fixed point $-\frac{\gamma}{2}$ of T at its branch points. Therefore, estimating the counting function $N(r, R)$ of the branch points of the Riemann surface R of $w(z)$, we have

$$N(r, R) \leqq N\left(r, -\frac{\gamma}{2}, w\right) \leqq T(r, w) + 0\,(1) .$$

According to the second fundamental theorem of Selberg, we have

$$\sum_a \delta(a, w) \leqq 2 + \varlimsup_{r \to \infty} \frac{N(r, R)}{T(r, w)} \leqq 3.$$

This is a contradiction and the case cannot occur. Q. E. D.

Theorem B (Ozawa [3]). Let R be a 2-sheeted covering surface of $|z| < \infty$ and P(R) the Picard constant of R. If $P(R) = 4$, then, R is defined by the algebroid function $w^2 = (e^{H(z)} - A)(e^{H(z)} - B)$, where $H(z)$ is an entire function and A and B are constants.

Proof. Let $u(z)$ be a meromorphic function on R with the irreducible defining equation $F(z,u) = f_0(z) u^2 + f_1(z) u + f_2(z) = 0$ and with four Picard exceptional values. According to the theorem of Cartan in § 1, we have $\lambda = 1$ for $u(z)$. Thus, as in the case of Theorem A, We may consider the following two cases.

1). If $f_2 = \alpha f_0 + \beta f_1$, we may assume four values to be a, $\frac{-\beta a - \alpha}{a + \beta}$, b and $\frac{-\beta b + \alpha}{b + \beta}$. we define $v(z) = \frac{u(z) - \omega_1}{u(z) - \omega_2}$, where ω_1 and ω_2 are the fixed points $-\beta \pm \sqrt{\beta^2 + \alpha}$. Then, we can see that $\{v(z)\}^2$ is a single-valued entire function with two Picard values $A = (\frac{a - \omega_1}{a - \omega_2})^2$ and $B = (\frac{b - \omega_1}{b - \omega_2})^2$. Setting $h(z) = B \frac{\{v(z)\}^2 - A}{\{v(z)\}^2 - B}$, we have an entire function with two Picard values 0 and ∞, that is, $h(z) = e^{H(z)}$. Solving this equation for $\{v(z)\}^2$, we have $\{v(z)\}^2 = B \frac{e^{H(z)} - A}{e^{H(z)} - B}$. This shows that the defining equation of R is $w^2 = (e^{H(z)} - A)(e^{H(z)} - B)$.

2). If $f_1 = \gamma f_0$, we may assume four Picard values to be a, $-a-\gamma$, b and $-b-\gamma$. Letting $v(z) = u(z) + \frac{\gamma}{2}$, $A = (a + \frac{\gamma}{2})^2$ and $B = (b + \frac{\gamma}{2})^2$, we have the same conclusion.

Q. E. D.

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