# A Remark on Complex Analytic Mappings

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# 複素解析写像に関する一注意

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Let R be a Riemnn surface of an algebroid function and M a Riemann surface of an algebraic function. The purpose of this paper is to give a proof of the second main theorem on complex analytic mappings of R into M by applying the Ahlfors theory of covering surfaces.

§1. Introduction. Let R be an n-sheeted covering surface of the complex plane $|z|<\infty$, which is a Riemann surface of an algebroid function, M an m-sheeted covering surface of the extended complex plane $|w|\leqq\infty$, which is a Riemann surface of an algebraic function of genus g, and $\varphi$ a complex analytic mapping of R into M. As for the Nevanlinna theory of such complex analytic mappings, the first main theorem is given in Hiromi-Mutō[1], in which some nonexsistence theorems on complex analytic mappings are also given. Further, the second main theorem and the deficiency relation are given in Noguchi[2] by using the differential geometric method.

On the other hand, concerning the Ahlfors theory of covering surfaces, Dufresnoy[3] and Tumura[4] develop the theory in the case of algebroid functions. The purpose of this paper is to give a proof of the second main theorem along this line. In § 2, we shall give a proposition on the Ahlfors theory of covering surfaces in the case of complex analytic mappings of R into M, and then in § 3, we shall give the proof applying this.

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§2. Let R. M. $\varphi$ be as in § 1. We consider the metric on M as is induced by $|w|\leqq\infty$. Let $R_r$ be the subregion of R over $|z|\leqq r$, and $M_r$ its image. According to the fundamental theorem of covering surfaces, we have

$$\rho \geqq (2g-2)S-hL,$$

where $\rho$, S and L are the Euler characteristic, the mean sheet number and the length of the relative boundary of $M_r$ and h is a constant depending only on M.

Here, we shall modify this inequality to a form applicable to our study. Let $a_1, \cdots\cdots, a_q$ be q points on M, $D_1, \cdots\cdots, D_q$ disjoint schlicht discs in M with their centers at $a_1, \cdots\cdots, a_q$ and $C_1, \cdots\cdots, C_q$ the circumferences of $D_1, \cdots\cdots, D_q$. If $M_r$ is decomposed by the cross cuts and the ring cuts over $C_1, \cdots\cdots, C_q$ into subregions, we call the subregions over $D_1, \cdots\cdots, D_q$ islands or peninsulas and the subregions over $M-\bigcup_{k=1}^{q} D_k$ lakes or seas respectively, according as the nonexistence or existence of the relative boundaries. Now, denoting by $p(D_k)$ the number of islands in $M_r$ over $D_k$, we have the following

PROPOSITION.

$$\sum_{k=1}^{q} p(D_k)+\rho \geqq (2g-2+q)S-hL.$$

Proof. Let $M_r$ be decomposed by the cross cuts and the ring cuts over $C_1, \cdots\cdots, C_q$ into subregions. We divide the islands, the peninsulas, the lakes and the seas further into two classes according as their Euler characteristics are nonnegative or equal to $-1$ and denote these classes $I_+$, $I_-$, $P_+$, $P_-$, $L_+$, $L_-$ and $S_+$, $S_-$ respectively, where the class $L_-$ is empty because of the Hurwitz formula. Now, estimating the Euler characteristic of each of those subregions, we have

$$\rho_i \geqq -1, \tag{1}$$
$$-1 = -1, \tag{2}$$
$$\rho_p \geqq 0, \tag{3}$$
$$-1 = -1, \tag{4}$$
$$\rho_l \geqq (2g-2+q)\, S_l, \tag{5}$$
$$\rho_s \geqq (2g-2+q)\, S_s - hL_s, \tag{6}$$
$$-1 = -1, \tag{7}$$

where $\rho_i$, $-1$, $\rho_p$, $-1$, $\rho_l$, $\rho_s$, $-1$ on the left sides are the Euler characteristics of the subregions in the classes $I_+$, $I_-$, $P_+$, $P_-$, $L_+$, $S_+$, $S_-$ respectively, $S_l$ and $S_s$ on the right sides are the mean sheet numbers of the subregions in the classes $L_+$ and $S_+$, and $L_s$ is the length of the relative boundary of the subregion in the class $S_+$. Here, (1), (2), (3), (4) and (7) are obvious and inequalities (5) and (6) are consequences of the fundamental theorem of covering surfaces. The following inequality (8) is also a consequence of the fundamental theorem of covering surfaces:

$$0 \geqq (2g-2+q)\, S_s' - hL_s', \tag{8}$$

where $S_s'$ and $L_s'$ are the mean sheet number and the length of the relative boundary of the subregion in the class $S_-$. Now, we add these inequalities (1) to (8) for all the subregions side by side, and then add N, the number of the cross cuts, to the both sides of the resulting inequality. If a region F is decomposed into two subregions, $F_1$ and $F_2$ by some ring cuts and n cross cuts, we have $\rho(F)=\rho(F_1)+\rho(F_2)+n$, where $\rho(F)$, $\rho(F_1)$ and $\rho(F_2)$ are the Euler characteristics of the regions F, $F_1$ and $F_2$. Hence, the left side is equal to $\rho$. As for the right side, we consider as follows. We decompose $M_r$ by only ring cuts into subregions and then these subregions by cross cuts, where the subregions without relative boundaries leave as they are. We denote the subregions with relative bondaries by $F_1, \cdots\cdots, F_l$, which are decomposed by the $N_1 \cdots\cdots, N_l$ $(N_1+\cdots\cdots+N_l=N)$ cross cuts. Then, each $F_k$ is decomposed into at most $N_k+1$ subregions, among which at least one is not simply connected, that is, has a nonnegative characteristic. So that, the sum of N and $-1$'s of (4) and (7) is nonnegative. Further, the sum of $-1$'s of (1) and (2) is the minus of the number of the islands, so that, is equal to $-\sum_{k=1}^{q} p(D_k)$. Consequently, denoting by $\bar{S}$ and $\bar{L}$ the mean sheet number and the length of the relative boundary of $M_r$ over $M-\bigcup_{k=1}^{q} D_k$, the right side is estimated from below at $-\sum_{k=1}^{q} p(D_k) + (2g-2+q)\,\bar{S}-h\bar{L}$. Therefore, we have

$$\sum_{k=1}^{q} p(D_k) + \rho \geqq (2g-2+q)\,\bar{S}-h\bar{L}.$$

Here,we have $|S-\bar{S}| \leqq hL$ by the first covering theorem, so that, $\bar{S} \geqq S-hL$. Clearly $\bar{L} \leqq L$ and we obtain the proposition.

§ 3. Let R, M, $\varphi$ be as in § 1. We denote by $p_R$ and $p_M$ the projection of R onto $|z|<\infty$ and the projection of M onto $|w| \leqq \infty$, and by $\tilde{\varphi}$ the algebroid function $p_M \circ \varphi$. First, following Hiromi-Mutō[1], we shall introduce the counting function and the proximity function of $\varphi$. Let a be a point of M of order $\lambda_a - 1$, and c a point of R of order $\lambda_c - 1$ such that $\varphi(c) = a$. Then, if we set $p_M(a) = w_o$ and $p_R(c) = z_o$, the algebroid function $\tilde{\varphi}$ can be represented in the neighborhood of $z_o$ as

$$w = w_o + \gamma_\tau \left(\sqrt[\lambda_c]{z-z_0}\right)^\tau + \cdots\cdots \qquad (\gamma_\tau \neq 0).$$

We set $n(r, a, \varphi) = \Sigma\tau$, where the summation $\Sigma$ is taken for all c such that $\varphi(c)=a$ and $|p_R(c)| \leqq r$, and define the counting function as follows:

$$N(r,a,\varphi) = \frac{1}{n\lambda_a}\left\{\int_0^r \frac{n(t,a,\varphi)-n(o,a,\varphi)}{t}dt + n(o,a,\varphi)\log r\right\}.$$

Let $K_a$ be the schlicht disc in M with the center a and the radius $\delta_a$. Setting

$$U_a(\alpha) = \begin{cases} \dfrac{1}{\lambda_a}\log\dfrac{\delta_a}{|w-w_o|}, & \alpha \in K_a \quad (P_M(\alpha)=w), \\ 0, & \alpha \notin K_a, \end{cases}$$

we define the proximity function as follows:

$$m(r,a,\varphi) = \frac{1}{2n\pi}\int_{\Gamma_r} U_a(\varphi(\gamma))\,d\theta,$$

where $\Gamma_r$ is the circumference of $R_r$ and $p_R(\gamma)=re^{i\theta}$. In these circumstances, $m(r,a,\varphi)+N(r,a,\varphi)$ is independent of a except for a bounded term. Namely it is known that the following theorem holds:

FIRST MAIN THEOREM (Hiromi-Mutō[1]).

$$m(r,a,\varphi) + N(r,a,\varphi) = \frac{1}{m}T(r,\tilde{\varphi})+O(1).$$

Here, $T(r,\tilde{\varphi})$ is the characteristic function of the algebroid function $\tilde{\varphi}$ in the sense of Selberg[5]. We define the characteristic function of $\varphi$ by

$$T(r,\varphi) = \frac{1}{m}T(r,\tilde{\varphi}).$$

Now, we shall give a proof of the second main theorem by using the proposition in § 2. Denoting by N(r,R) the counting function of the branch points of R, we have the following

SECOND MAIN THEOREM (cf. Noguchi[2]).

$$(2g-2+q)\,T(r,\varphi) \leqq \sum_{k=1}^{q} N(r,a_k,\varphi) + N(r,R) + O(\sqrt{T(r,\varphi)}\log T(r,\varphi)), \qquad r\notin E,$$

where E satisfies $\int_E \frac{1}{r\log r}dr<\infty$.

Proof. Let $S(t)$, $\rho(t)$ and $L(t)$ be the quantities S, $\rho$, L for $M_t$ in § 2. Then, the proposition in § 2 is

$$(2g-2+q)\,S(t) \leqq \sum_{k=1}^{q} p(D_k) + \rho(t) + hL(t).$$

We divide the both sides by nt, and integrate them from $r_0$ to r. It is well-known that

$$\frac{1}{n}\int_{r_0}^{r}\frac{S(t)}{t}dt = T(r,\varphi) + O(\log r).$$

Since $p(D_k) \leqq n(t,a_k,\varphi)$, we have

$$\frac{1}{n}\int_{r_0}^{r}\frac{p(D_k)}{t}dt \leqq N(r,a_k,\varphi) + O(\log r).$$

Further, we have $\rho(t) = n(t,R) - n$ by the Hurwitz formula, where $n(t,R)$ is the sum of orders of the branch points of $R_t$. Thus, we have

$$\frac{1}{n}\int_{r_0}^{r}\frac{\rho(t)}{t}dt = N(r,R)+O(\log r).$$

Last, following Dufresnoy[3] and Dinghas[6], we have

$$\frac{1}{n}\int_{r_0}^{r}\frac{L(t)}{t}dt = O(\sqrt{T(r,\varphi)}\log T(r,\varphi)), \qquad r\notin E,$$

in the usual manner. Combining these equalities and inequalities, we obtain the theorem.

The rest of the paper is concerned with the deficiency relation. For $a\in M$, we define the deficiency as

$$\delta(a,\varphi) = 1-\overline{\lim_{r\to\infty}}\frac{N(r,a,\varphi)}{T(r,\varphi)}.$$

Now, let $\varphi$ be the complex analytic mapping of R into M such that the proper existence domain of the algebroid function $\tilde{\varphi}$ coincides with R. Then, according to the branch point theorem of Ullrich[7], we have

$$N(r,R)\leqq(2n-2)mT(r,\varphi).$$

Therefore, for such $\varphi$, we have the following

DEFICIENCY RELATION (cf. Noguchi[2]).

$$\sum_{k=1}^{q}\delta(a_k,\varphi) \leqq (2-2g) + 2(n-1)\,m.$$

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