A Construction of Degenerate CM-types

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退化CM型の構成

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We show a method to construct degenerate CM-types.

These CM-types give examples of abelian varieties whose Hodge cycles are not generated by the divisor classes.

Rank of CM-type

Let $p=2d+1$ be an odd prime number. We denote by G the cyclic group of nonzero residue classes mod p. A subset S of G is called *CM-type* if $S \cup -S=G$ and $S \cap -S=\phi$, where $-S=\{-a \mid a \in S\}$. If $\sigma S=S \Longleftrightarrow \sigma=1$, then S is called *simple*. For example, $\{1, 2, \ldots, \frac{p-1}{2}\}$ is a simple CM-type.

For a CM-type S, let $\Phi_S : Z[G] \to Z[G]$ be a homomorphism defined by

$$\Phi_S(x) = \sum_{\sigma \in S} \sigma x.$$

We call the rank of the image of Φ_S, the *rank of a CM-type* S. It is easily seen that *rank* $S \leqq d+1$. When *rank* $S=d+1$, the CM-type S is said to be *nondegenerate* (cf. Kubota[1)]). Non-simple CM-type must be degenerate.

Hodge cycle

A subset $T \subset G$ is called a *Hodge cycle* when $\#(\sigma T \cap S)$ does not depend on $\sigma \in G$. Then $\# T$ must be even. For any non empty $T' \subset G$, $T=T' \cup -T'$ is a Hodge cycle. Such Hodge cycles are characterized by $T=-T$. There are possibly Hodge cycles which do not satisfy this condition. They are called *exceptional*. If there is an exceptional Hodge cycle, then the CM-type must be degenerate.

EXAMPLE (Serre). $p=19$, $S=\{1, 3, 4, 5, 6, 7, 8, 10, 17\}$, $T=\{1, 2, 3, 7, 11, 14\}$. Then T is an exceptional Hodge cycle for S.

REMARK. It is known that simple degenerate CM-type has always an exceptional Hodge cycle (Lenstra). But, for more general CM-types (which are not considered in this note), this is an open problem.

Construction

We now show a method to construct degenerate CM-types.

ASSUMPTION. Let $p=2mn+1$ with odd integers $m, n>1$.

Let H be the subgroup of G with order m, and σ_0 be a gererator (a primitive root) of G. Then the coset $\sigma_0 H$ is a generator of the cyclic group G/H. For a decomposition $m=r+s$ $(r, s \geqq 1)$, pick up r elements of H, and s elements of $\sigma_0 H$, r elemests of $\sigma_0^2 H, \ldots$, r elements of $\sigma_0^{n-1} H$. We denote the set of these elements by S_1. Then $S=-((H \cup \sigma_0 H \cup \ldots \cup \sigma_0^{n-1} H) \setminus S_1) \cup S_1$ is a CM-type, and $T=H \cup \sigma_0 H$ is an exceptional Hodge cycle for S.

EXAMPLE. $p=31$, $m=3$, $n=5$, $\sigma_0=3$, $r=2$, $s=1$, $H=\{1, 5, 25\}$, $\sigma_0 H=\{3, 13, 15\}$. If we pick up two elements of H, and one element of $\sigma_0 H$, . . . as above, then we get a degenerate CM-type. For example, $S=\{1, 5, 13, 9, 14, 24, 19, 2, 6, 28, 16, 23, 4, 20, 21\}$ is degenerate (and simple), and $T=\{1, 5, 25, 3, 13, 15\}$ is an exceptional Hodge cycle for S.

Background

The notion of CM-type has arisen from the theory of complex multiplication of abelian varieties. In our case, the field of complex multiplication is the field of p-th root of unity (cf. Shimura-Taniyama[2)]).

Exceptional Hodge cycles correspond to Hodge cycles that are not generated by the intersections of divisors on the variety (cf. Ribet[3)]). Our method can be applicable to more general CM-types.

Addendum. B. Dodson (Trans. AMS. 283, n°1, 1984) has given similar and more general results. Our method is elementary and explicit.

References

1) Kubota T.: On the field extension by complex multiplication, Trans. Amer. Math. Soc., 118, n°6, 113-122, 1965.
2) Shimura G. and Taniyama Y.: Complex multiplication of Abelian Varieties and its Application to Number Theory, Publ. Math. Soc. Japan, n°6, 1961.
3) Ribet K. A.: Hodge classes on certain types of abelian varieties, Amer. J. Math., 105, 523-538, 1983.

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