愛知工業大学研究報告
第３３号A 平成10年

On the Mandelbrot Set of $w = cz(3 - z^2) + 1$

$w = cz(3 - z^2) + 1$ の Mandelbrot 集合について

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The Mandelbrot set of the quadratic polynomial $p_c(z) = z^2 + c$ is the set of those values c such that the iteration sequence $\{p_c^n(0)\}$ of the finite critical point 0 of $p_c(z)$ is bounded. Similarly, we can define the Mandelbrot set of the cubic polynomial $q_c(z) = cz(3 - z^2) + 1$ with two finite critical points 1 and -1. And investigating this Mandelbrot set, we can obtain some examples of Julia sets which are disconnected, but need not be totally disconnected.

§1. Introduction.

Let $p_c(z) = z^2 + c$ be a quadratic polynomial with a complex parameter c. The Mandelbrot set $\mathcal{M}$ of $p_c(z)$ is defined by

$$\mathcal{M} = \widehat{C} - \{c | \lim_{n\to\infty} p_c^n(0) = \infty\},$$

where $\widehat{C}$ is the extended complex plane and ∞ is the point at infinity. The Mandelbrot set is the set of those values c such that the iteration sequence $\{p_c^n(0)\}$ of the finite critical point 0 of $p_c(z)$ is bounded and can be written more precisely as

$$\mathcal{M} = \bigcap_{n=1}^{\infty} \{c | \, |p_c^n(0)| \leq 2\}.$$

On the other hand, the Mandelbrot set is the set of those values c such that the corresponding Julia set $\mathcal{J}_c$ of $p_c(z)$ is connected. Further, for the value c of $\widehat{C} - \mathcal{M}$, the corresponding Julia set $\mathcal{J}_c$ is totally disconnected.

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The similar situation occurs in the case of those polynomials with one finite critical point. The investigation of the Mandelbrot set of $p_{c,n}(z) = z^n + c$ is given in [4] along this line. In the case of those polynomials with two or more finite critical points, the situation is much more complicated. In this paper, we consider the polynomial $q_c(z) = cz(3 - z^2) + 1$ with two finite critical points 1 and -1. In §2, we define the Mandelbrot set M of $q_c(z)$ as the intersction of the sets M_1 and M_{-1} which are the sets of those values c such that the iteration sequences $\{q_c^n(1)\}$ and $\{q_c^n(-1)\}$ of the finite critical points 1 and -1 of $q_c(z)$ are bounded respectively. We shall also give the computer graphics of M_1, M_{-1} and M. In §3, we shall investigate the Julia set $\mathcal{J}_c$ of $q_c(z)$. We shall give the computer graphic of the Julia set $\mathcal{J}_{1+0.1i}$ which is disconnected, but is not totally disconnected. Finally, in §4, we shall give some problems left open.

§2. The Mandelbrot set of $q_c(z)$.

Let $q_c(z) = cz(3-z^2)+1$ be the cubic polynomial of a complex parameter c. The critical points of $q_c(z)$ are given by the equation $q_c'(z) = 3c(1 - z^2) = 0$, and are 1 and -1. We define the sets M_1 and M_{-1} as the sets of those values c such that the iteration sequences $\{q_c^n(1)\}$ and $\{q_c^n(-1)\}$ are bounded respectively. That is,

$$M_1 = \widehat{C} - \{c \mid \lim_{n\to\infty} q_c^n(1) = \infty\}$$

and

$$M_{-1} = \widehat{C} - \{c \mid \lim_{n\to\infty} q_c^n(-1) = \infty\}.$$

As in the case of the Mandelbrot set $\mathcal{M}$ of $p_c(z)$, we have the following precise representations of M_1 and M_{-1}.

Theorem 1. M_1 and M_{-1} are closed sets contained in the disk $\{|c| \le \frac{3}{2}\}$ and can be written as

$$M_1 = \bigcap_{n=1}^{\infty} \left\{c \mid |q_c^n(1)| \le \sqrt{\frac{2}{|c|} + 3}\right\}$$

and

$$M_{-1} = \bigcap_{n=1}^{\infty} \left\{c \mid |q_c^n(-1)| \le \sqrt{\frac{2}{|c|} + 3}\right\},$$

where $c = 0$ is considered to be contained in M_1 and M_{-1}.

Proof. We prove the theorem in the case of M_1. Let c be the value satisfying $|c| > \frac{3}{2}$. Setting $|c| = \frac{3}{2} + \delta$ $(\delta > 0)$, we have

$$|q_c(1)| = |2c + 1| \ge 2|c| - 1 > 2 + \frac{3}{2}\delta.$$

Further, by induction, we have

$$|q_c^n(1)| > 2 + \left(\frac{3}{2}\right)^n \delta.$$

Therefore, $c \notin M_1$ and $M_1 \subset \left\{|c| \leq \frac{3}{2}\right\}$.

Next, let c be the value satisfying $|q_c^n(1)| > \sqrt{\frac{2}{|c|} + 3}$. Setting $|q_c^n(1)| = \sqrt{\frac{2}{|c|} + 3 + \delta}$ $(\delta > 0)$, we have

$$\begin{aligned} |q_c^{n+1}(1)| &= |\ c\ q_c^n(1)\{3 - (q_c^n(1))^2\} + 1| \\ &\geq |c||q_c^n(1)|(|q_c^n(1)|^2 - 3) - 1 \\ &> |c|\left(\sqrt{\frac{2}{|c|} + 3 + \delta}\right)\frac{2}{|c|} - 1 \\ &> \sqrt{\frac{2}{|c|} + 3 + 2\delta}. \end{aligned}$$

Proceding by induction, we have

$$\left|q_c^{n+k}(1)\right| > \sqrt{\frac{2}{|c|} + 3 + 2^k\delta}.$$

Therefore, $c \notin M_1$ and $M_1 \subset \left\{c \mid |q_c^n(1)| \leq \sqrt{\frac{2}{|c|} + 3}\right\}$. This inclusion is valid for all prositive integers n, so that we have

$$M_1 \subset \bigcap_{n=1}^{\infty} \left\{c \mid |q_c^n(1)| \leq \sqrt{\frac{2}{|c|} + 3}\right\}.$$

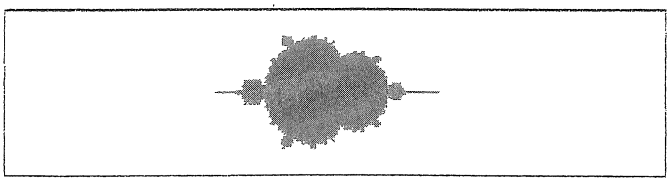
Since the converse inclusion is obvious, we obtain

$$M_1 = \bigcap_{n=1}^{\infty} \left\{c \mid |q_c^n(1)| \leq \sqrt{\frac{2}{|c|} + 3}\right\}.$$

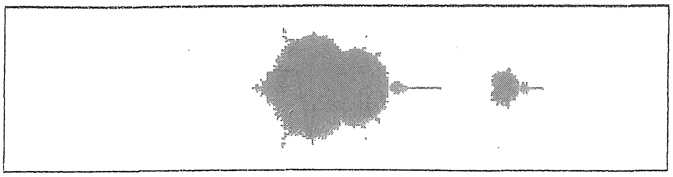
We can see that M_1 is closed, as the intersection of the closed sets is also closed.

The case is the same for M_{-1} and we obtain the theorem. Q.E.D.

Theorem 1 sugests a simple algorithm to give the computer graphics of M_1 and M_{-1}. The following graphics are those given by this algorithm.



M_1 ($|\text{Rec}| \leq 2, |\text{Imc}| \leq 0.5$)



M_{-1} ($|\mathrm{Re}c| \leq 2, |\mathrm{Im}c| \leq 0.5$)

According to the computer graphic of M_1, the number of the connected components of M_1 seems to be two. But by magnifying this graphic, we can see that there exist many other connected components of M_1.

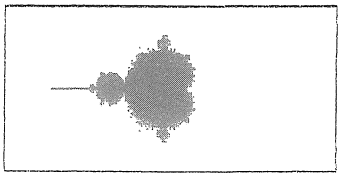
The 2-cycles of M_1 are given by the equation

$$(2c+1)(2c^2+2c-1)=0.$$

And the 3-cycles of M_1 are given by the equations

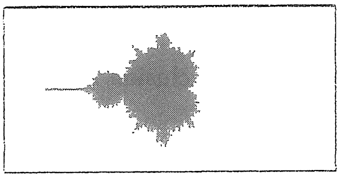
$$2c(2c+1)(2c^2+2c-1)-1=0,$$
$$2c(2c+1)(2c^2+2c-1)-1+\sqrt{3}=0,$$
$$2c(2c+1)(2c^2+2c-1)-1-\sqrt{3}=0.$$

Among these cycles, we shall give the magnifications of M_1 near the 2-cycle $c=-1.36602\cdots$ and the 3-cycle $c=-1.490597\cdots$.



magnification of M_1

($|\mathrm{Re}c+1.36602| \leq 0.005, |\mathrm{Im}c| \leq 0.0025$)



magnification of M_1

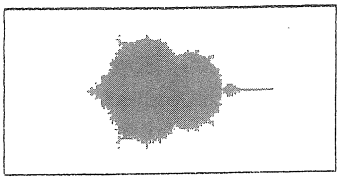
($|\mathrm{Re}c+1.490597| \leq 0.00002, |\mathrm{Im}c| \leq 0.00001$)

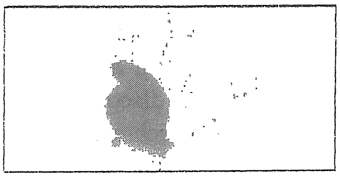
Continuing this process, we can find the sequence of the connected components of M_1 tending to the point $c=-\frac{3}{2}$. Therefore $\widehat{C}-M_1$ is of infinite connectivity. The case is the same for M_{-1}.

The Mandelbrot set of $q_c(z)$ is to be defined as the set of those values c such that the Julia set $\mathcal{J}_c$ of $q_c(z)$ is connected. Therefore, we define the Mandelbrot set M of $q_c(z)$ by

$$M=M_1\cap M_{-1}.$$

The computer graphic of M and the magnification of M near the point $c=0.2957+0.2369\,i$ are the following.





M magnification of M

$(|\text{Re}c| \leq 1, |\text{Im}c| \leq 0.5)$ $(|\text{Re}c - (0.2957 + 0.2369i)| \leq 0.005, |\text{Im}c| \leq 0.0025)$

Accoding to Theorem 1, M is contained in the disk $\{|c| \leq \frac{3}{2}\}$. M is also consider to be disconnected, which is contrary to the case of the Mandelbrot set $\mathcal{M}$ of $p_c(z) = z^2 + c$.

§3. The Julia set of $q_c(z)$.

Let H be the set of values c such that there exists an attracting or super-attracting fixed point of $q_c(z)$. Concerning H, we have the following theorem.

Theorem 2. H is a domain bounded by the algebraic curves and is represented by

$$H = \left\{c \mid 4c^3 - 4c^2 + (-3\lambda^2 + 2\lambda)c - \lambda(\lambda - 1)^2 = 0, \ |\lambda| < \frac{1}{3}\right\}.$$

Proof. According to the conditions on H, we have

$$q_c(z) = cz(3 - z^2) + 1 = z$$

and

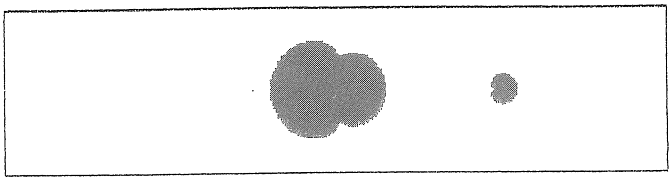
$$|q_c'(z)| = |3c(1 - z^2)| < 1.$$

Setting $\lambda = c(1 - z^2)$, we have $z = \frac{1}{1 - 2c - \lambda}$. Therefore, from these equations, we have

$$4c^3 - 4c^2 + (-3\lambda^2 + 2\lambda)c - \lambda(\lambda - 1)^2 = 0$$

and $|\lambda| < \frac{1}{3}$. Q.E.D.

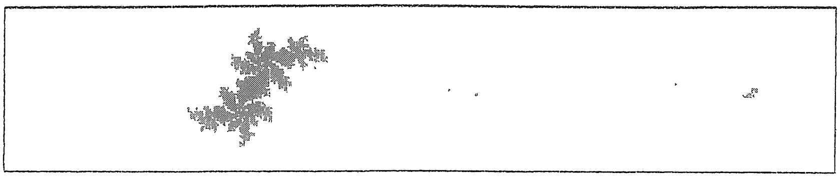
The computer graphic of H is shown in the following.



H $(|\text{Re}c| \leq 2, |\text{Im}c| \leq 0.5)$

We can see that H is the union of the component of the interior of M containing $c = 0$ and the component of the interior of M_{-1} containing $c = 1$.

The point $c = 1$ is contained in M_{-1} and is not contained in M_1. So that, the Julia set $\mathcal{J}_1$ of $q_1(z) = z(3 - z^2) + 1$ is disconnected but is not totally disconnected. We give the computer graphic of $\mathcal{J}_{1+0.1i}$ which shows the fructal structure of Julia sets much better.



$\mathcal{J}_{1+0.1i}$ ($|\mathrm{Re}c| \leq 2, |\mathrm{Im}c| \leq 0.5$)

§4. Problems.

The investigation of the Mandelbrot set of $q_c(z) = cz(3 - z^2) + 1$ in §**2** and §**3** leads us to some problems left open. Among these problems, we give the following two problems.

(1) Does the set M have uncountably many connected components?

(2) How can one expain on the semi-fractal sets appearing in the magnification of M? Both problems seem to be difficult.

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(受理 平成10年 3 月20日)