愛知工業大学研究報告
第 37 号 A 平成 14 年

Polynomial solutions to boundary-value problems of the heat equation

熱方程式の境界値問題に対する多項式解

Dedicated to Professor Masayuki Itô in honour of his sixtieth birthday

Gou NAKAMURA† and Noriaki SUZUKI††
中村 豪 鈴木紀明

Abstract. In this paper we shall determine a polynomial $\psi(x,t)$ of degree at most 3 such that for any polynomial $f(x,t)$ there exists a heat polynomial $u(x,t)$ which equals $f(x,t)$ on the curve $\psi(x,t)=0$.

1 Introduction

Let $\mathcal{P}$ be the set of polynomials in two variables x and t with real coefficients, and $\mathcal{P}_m$ the subset of $\mathcal{P}$ of degree at most m. The heat operator L is defined in $\mathbb{R}^2$ by

$$L[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}.$$

Let $\mathcal{HP}$ be the set of heat polynomials in $\mathcal{P}$.

Basic Problem. Let $\psi \in \mathcal{P}$. Then for any $f \in \mathcal{P}$, is there a polynomial solution $u \in \mathcal{P}$ satisfying the following (1)-(2)?

$$L[u] = 0 \text{ in } \mathbb{R}^2, \tag{1}$$
$$u(x,t) = f(x,t) \text{ on } \psi(x,t)=0. \tag{2}$$

Definition 1.1 *A polynomial ψ is said to be square-free if*

(i) *ψ is minimal, that is, ψ has no repeated factors such as $p(x,t)^m$ $(m \geq 2)$, and*

(ii) *for each irreducible factor ψ_i with real coefficients of ψ, $\psi_i = 0$ has infinitely many points.*

We have the following algebraic result [1].

Theorem 1.2 *Let ψ be square-free, and $f \in \mathcal{P}$. If $u \in \mathcal{P}$ satisfies* (2)*, then there exists $g \in \mathcal{P}$ such that $u - f = \psi g$.*

Hence we can say that the Basic Problem is to find ψ such that

$$\mathcal{HP} + \psi\mathcal{P} = \mathcal{P}.$$

Theorem 1.3 *Let ψ be square-free, and $m \geq 2$. For any $f \in \mathcal{P}_m$, if there exists $u \in \mathcal{P}_m$ satisfying* (1)-(2)*, then* $\deg \psi = 1$.

Proof. Suppose that $\psi \in \mathcal{P}_k$, $k \geq 1$. Consider a linear mapping T from $\mathcal{P}_{m-k}$ onto $\mathcal{P}_{m-1}$ as follows:

$$\begin{array}{ccccc} T & : & \mathcal{P}_{m-k} & \to & \mathcal{P}_{m-1}. \\ & & \Psi & & \Psi \\ & & g & \mapsto & L[\psi g] \end{array}$$

†愛知工業大学 基礎教育センター（豊田市）
††名古屋大学大学院 多元数理科学研究科（名古屋市）

We shall show that T is surjective. For any $h \in \mathcal{P}_{m-1}$, there exists $f \in \mathcal{P}_m$ such that $L[f] = h$ because $\{L[f]\ ;\ f \in \mathcal{P}_m\} = \mathcal{P}_{m-1}$. From our assumption there exists a solution $u \in \mathcal{P}_m$ for f. By Theorem1.2, we have $g \in \mathcal{P}_{m-k}$ such that $u - f = -\psi g$. Then it follows $T(g) = L[\psi g] = L[f-u] = L[f] = h$. Thus we see that T is surjective. The surjectivity of T gives

$$\dim \mathcal{P}_{m-k} = {}_{m-k+2}C_2 \geq {}_{m+1}C_2 = \dim \mathcal{P}_{m-1}.$$

Therefore $k \leq 1$. □

Put

$$v_n(x,t) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{t^k}{k!} \frac{x^{n-2k}}{(n-2k)!} \quad (n = 0, 1, \ldots).$$

Then each $v_n(x,t)$ is a heat polynomial.

Lemma 1.4 *The set $\{v_n(x,t)\}$ is a basis for $\mathcal{HP}$.*

Proof. A polynomial $p(x,t)$ of degree n is of the form

$$p(x,t) = ax^n + \sum_{j=1}^{n} a_j x^{n-j} t^j + (\text{terms of degree} \leq n-1).$$

If $p(x,t)$ is a heat polynomial, then

$$L[p] = -\sum_{j=1}^{n} j a_j x^{n-j} t^{j-1} + (\text{terms of degree} \leq n-2) = 0.$$

Hence $ja_j = 0$ for $j = 1, \ldots, n$ and

$$p(x,t) = ax^n + p_{n-1}(x,t) \quad (\deg p_{n-1} \leq n-1).$$

Since $v_n(x,t) = x^n + (\text{terms of degree} \leq n-1)$, $q_{n-1}(x,t) = p(x,t) - av_n(x,t)$ is a heat polynomial of degree at most $n-1$. By the induction we see that any heat polynomial is constructed by $\{v_n\}$. Uniqueness of the linear conbination follows from the linear independence of $\{v_n\}$. □

Lemma 1.5 *Let $\psi \in \mathcal{P}$ of $\deg \psi \geq 2$. If the Basic Problem holds for ψ, then the variable of the highest degree term of ψ is only x.*

Proof. If the Basic Problem holds for ψ, then Theorem 1.3 implies that for some $f \in \mathcal{P}$ the solution u satisfies $\deg f < \deg u$. Since the solution u is of the form $u = f + \psi g$ by Theorem 1.2, we have $\deg u = \deg \psi g$. By Lemma 1.4, the highest degree term of a heat polynomial u is a polynomial of x, so is that of ψg. Hence the variable of highest degree term of ψ is only x. □

2 Linear equations

Theorem 2.1 *Suppose that ψ has $\deg \psi = 1$, that is, the equation $\psi(x,t) = 0$ defines a line $ax + bt + c = 0$. Then the Basic Problem is solved according to the gradient of the line as follows:*

(i) *if $b \neq 0$, then there exists a unique solution,*

(ii) *if $b = 0$, then there exists a non-unique solution.*

Proof. Since the set of heat polynomials is invariant with respect to any parallel translation, we can take $\psi(x,t) = 0$ as $ax + bt = 0$.

(i) If $b \neq 0$, we can take $\psi = 0$ as $t = ax$. Substitute it for each $v_n(x,t)$, then

$$\begin{aligned} v_n(x, ax) &= n! \sum_{k=0}^{[\frac{n}{2}]} \frac{a^k x^{n-k}}{k!(n-2k)!} \\ &= x^n + (\text{lower degree terms}). \end{aligned}$$

Therefore for any $f \in \mathcal{P}$ of degree N, there exist $c_0, c_1, \ldots, c_N \in \mathbb{R}$ such that

$$f(x, ax) = \sum_{n=0}^{N} c_n v_n(x, ax),$$

where $c_0, c_1, \ldots, c_N$ are uniquely determined. Put $u(x,t) = \sum_{n=0}^{N} c_n v_n(x,t)$, then we see that $u(x,t)$ is a unique solution to the Basic Problem.

(ii) If $b = 0$, we can take $\psi = 0$ as $x = 0$. Substitute it for each $v_n(x,t)$, then

$$v_{2n}(0,t) = \frac{(2n)!}{n!} t^n \quad \text{and} \quad v_{2n+1}(0,t) = 0.$$

Therefore for any $f \in \mathcal{P}$ of degree N, there exist $c_0, c_1, \ldots, c_N \in \mathbb{R}$ such that

$$f(0,t) = \sum_{n=0}^{N} c_n v_{2n}(0,t).$$

Put $u(x,t) = \sum_{n=0}^{N} c_n v_{2n}(x,t)$, then we see that $u(x,t)$ is a solution to the Basic Problem. Since $v_{2n+1}(0,t) = 0$, $u(x,t) + v_{2n+1}(x,t)$ is also a solution. Hence the uniqueness of the solution does not hold. □

3 Quadratic equations

Theorem 3.1 *Let ψ be a square-free polynomial of* $\deg \psi = 2$. *Then the Basic Problem is answered affirmatively if and only if $\psi(x,t) = 0$ is the following:*

(i) *two lines parallel to the t-axis, or*

(ii) *parabolas obtained by parallel translations of $x^2 = 4pt$ $(p > 0)$, or*

(iii) *parabolas obtained by parallel translations of $x^2 = 4pt$ $(p < 0)$ such that $\sqrt{-p}$ is not a zero point of any Hermite polynomials.*

Furthermore, the solution u is unique in each case.

Proof. Every quadratic polynomial $\psi(x,t)$ is of form $Ax^2 + Bxt + Ct^2 + Dx + Et + F = 0$. If the Basic Problem holds for $\psi(x,t)$, then it follows that $B = C = 0$ from Lemma 1.5. Since ψ is quadratic, we have $A \neq 0$ and assume that $A = 1$. Furthermore, translating the equation by $x \to x - D/2$, we can take $\psi(x,t) = 0$ as $x^2 + bt + c = 0$.

(i) If $b = 0$, we have $\psi(x,t) = x^2 + c = 0$ and $c < 0$ because ψ is square-free. In this case it is a pair of lines parallel to the t-axis.

Any polynomial $f(x,t)$ is reduced to the form $f(x,t) = f_1(t) + x f_2(t)$ on $x^2 + c = 0$. Also, $\{v_n(x,t)\}$ is reduced to the form

$$\begin{aligned} v_{2n}(x,t) &= (2n)! \sum_{k=0}^{n} \frac{t^k}{k!} \frac{(-c)^{n-k}}{(2n-2k)!} \\ v_{2n+1}(x,t) &= (2n+1)! \sum_{k=0}^{n} \frac{t^k}{k!} \frac{(-c)^{n-k}}{(2n+1-2k)!} x \end{aligned}$$

on $x^2+c=0$. Then there exist $c_0, c_1, \ldots, c_N$ and $d_0, d_1, \ldots, d_M$ such that

$$
\begin{aligned}
f_1(t) &= \sum_{n=0}^{N} c_n v_{2n}(x,t) \text{ and} \\
x f_2(t) &= \sum_{n=0}^{M} d_n v_{2n+1}(x,t)
\end{aligned}
$$

on $x^2+c=0$. Therefore

$$u(x,t) = \sum_{n=0}^{N} c_n v_{2n}(x,t) + \sum_{n=0}^{M} d_n v_{2n+1}(x,t)$$

is a solution. We shall show the uniqueness of the solution u. For $f(x,t) \equiv 0$, there exists a solution $u(x,t) = \sum_{n=0}^{N} c_n v_n(x,t)$. Then for any points (x,t) and $(-x,t)$ on $x^2+c=0$, $u(x,t)$ satisfies

$$0 = u(\pm x, t) = \sum_{n=0}^{[\frac{N}{2}]} c_{2n} v_{2n}(x,t) \pm \sum_{n=0}^{[\frac{N-1}{2}]} c_{2n+1} v_{2n+1}(x,t).$$

Hence $\sum_{n=0}^{[\frac{N}{2}]} c_{2n} v_{2n}(x,t) = 0$ and $\sum_{n=0}^{[\frac{N-1}{2}]} c_{2n+1} v_{2n+1}(x,t) = 0$ on $x^2+c=0$, and we have $c_n = 0$ $(n = 0, 1, \ldots, N)$.

If $b \neq 0$, then we can take $\psi(x,t) = 0$ as $x^2 + bt = 0$ by translating $t \to t - c/b$. Put $b = -4p$, then we have $x^2 = 4pt$. Substituting $t = x^2/(4p)$ for $\{v_n(x,t)\}$, we have

$$v_n\left(x, \frac{x^2}{4p}\right) = n! \sum_{k=0}^{[\frac{n}{2}]} \left(\frac{x^2}{4p}\right)^k \frac{x^{n-2k}}{k!(n-2k)!} = n! \sum_{k=0}^{[\frac{n}{2}]} \left(\frac{1}{4p}\right)^k \frac{x^n}{k!(n-2k)!}.$$

(ii) If $p > 0$, then the coefficient of x^n for $v_n(x, x^2/(4p))$ is non-zero. So that for any $f \in \mathcal{P}$, we can construct $f(x, x^2/(4p))$ by $\{v_n(x, x^2/(4p))\}$. Hence there exists a solution u and it is uniquely determined.

(iii) If $p < 0$, then the coefficient of x^m for $v_m(x, x^2/(4p))$ may be zero for some m. In case it happens, $f(x,t) = x^m$ cannot be constructed by $\{v_n(x, x^2/(4p))\}$. Since

$$v_n(x,t) = (-t)^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{-4t}}\right), \ t < 0,$$

where $H_n(x)$ denotes the Hermite polynomial of degree n, we have

$$v_n\left(x, \frac{x^2}{4p}\right) = \frac{x^n}{(2\sqrt{-p})^n} H_n(\sqrt{-p}).$$

Therefore $v_n(x, x^2/(4p)) \equiv 0$ if and only if $\sqrt{-p}$ is the zero point of $H_n(x)$. □

4 Equations of degree 3

Theorem 4.1 *Let ψ be a square-free polynomial of* $\deg \psi = 3$. *Then the Basic Problem is answered negatively.*

Proof. If the Basic Problem holds for $\psi(x,t)$, then it follows that $\psi(x,t) = Ax^3 + Bx^2 + Cxt + Dt^2 + Ex + Ft + G$ from Lemma 1.5. Then we can assume that $A = 1$ and that $B = 0$ by translating $x \to x - B/3$. So that $\psi = 0$ is reduced to $x^3 + Cxt + Dt^2 + Ex + Ft + G = 0$.

First, we shall show that $D = 0$. Suppose that $D \neq 0$. Since $\psi = 0$ is a quadratic equation of t, we have

$$t = \varphi(x) = \frac{1}{2D}\{-Cx - F \pm \sqrt{(Cx+F)^2 - 4D(x^3 + Ex + G)}\}$$

for sufficiently large $x > 0$ or small $x < 0$ according to $D < 0$ or $D > 0$, respectively. Then $\varphi(x) = O(x^{3/2})$ $(x \to \infty$ or $-\infty)$ and

$$v_n(x, \varphi(x)) = x^n + n(n-1)\varphi(x)x^{n-2} + O(x^{n-1}).$$

For $f(x,t) = x^2$, there exists a solution $u(x,t) = \sum_{n=0}^{N} c_n v_n(x,t)$, $c_N \neq 0$, so that

$$\begin{aligned} x^2 &= u(x, \varphi(x)) \\ &= \sum_{n=0}^{N} c_n v_n(x, \varphi(x)) \\ &= c_N x^N + c_N N(N-1)\varphi(x)x^{N-2} + O(x^{N-1}). \end{aligned}$$

Clearly $N > 2$. Since we can take $x \to \infty$ or $-\infty$ for (x,t) on $\psi(x,t) = 0$,

$$\frac{1}{x^{N-2}} = c_N + \frac{c_N N(N-1)\varphi(x)}{x^2} + O\left(\frac{1}{x}\right)$$

implies $c_N = 0$, which contradicts $c_N \neq 0$. Hence $D = 0$.

Next, we shall show that $C \neq 0$. Suppose that $C = 0$, then $x^3 + Ex + Ft + G = 0$. We consider this equation according to $F \neq 0$ or $F = 0$.

If $F \neq 0$, then by translating $t \to t - G/F$ we have $x^3 + Ex + Ft = 0$. Substitute $t = (x^3 + Ex)/(-F)$ for $v_n(x,t)$, then

$$\begin{aligned} v_n\left(x, \frac{x^3 + Ex}{-F}\right) &= n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(x^3 + Ex)^k x^{n-2k}}{(-F)^k k!(n-2k)!} \\ &= \frac{n!}{(-F)^{[\frac{n}{2}]}[\frac{n}{2}]!} x^{n+[\frac{n}{2}]} + \text{(lower degree terms)}. \end{aligned}$$

For $f(x,t) = x^2$, there exists a solution $u(x,t) = \sum_{n=0}^{N} c_n v_n(x,t)$, $c_N \neq 0$, so that

$$\begin{aligned} x^2 &= \sum_{k=0}^{N} c_n v_n\left(x, \frac{x^3 + Ex}{-F}\right) \\ &= c_N \frac{N!}{(-F)^{[\frac{N}{2}]}[\frac{N}{2}]!} x^{N+[\frac{N}{2}]} + \text{(lower degree terms)}. \end{aligned}$$

Consequently it follows that $2 = N + [N/2]$, which never occurs.

If $F = 0$, then $\psi(x,t) = x^3 + Ex + G$ is factorized to

$$\psi(x,t) = (x-a)(x-b)(x-c),$$

where $a, b, c \in \mathbb{R}$ are distinct because ψ is square-free. As we have seen in the quadratic cases, the solution of the Basic Problem is uniquely determined by two lines parallel to the t-axis. Hence it does not hold in the case of three parallel lines.

By translating $x \to x - F/C$ and $t \to t - 3F^2/C^3 - E/C$ for $x^3 + Cxt + Ex + Ft + G = 0$ $(C \neq 0)$, we take $\psi = 0$ as $x^3 + \alpha x^2 + Cxt + \beta = 0$. Then $\beta \neq 0$. In fact, if $\beta = 0$, then $x(x^2 + \alpha x + Ct) = 0$. For $f(x,t) = x^2 + \alpha x + Ct$, a heat polynomial u satisfying $u = f$ on the quadratic curve $x^2 + \alpha x + Ct = 0$ is only $u \equiv 0$ by Theorem 3.1. But u is not identically equal to f on the line $x = 0$. Hence $\beta \neq 0$.

Last, we shall show that the Basic Problem does not hold even if $\beta \neq 0$. Hence it never holds for any ψ of degree 3. Substitute $t = (x^3 + \alpha x^2 + \beta)/(-Cx)$ for $v_n(x,t)$, then

$$\begin{aligned} v_n\left(x, \frac{x^3 + \alpha x^2 + \beta}{-Cx}\right) &= n! \sum_{k=0}^{[\frac{n}{2}]} \frac{(x^3 + \alpha x^2 + \beta)^k x^{n-2k}}{(-Cx)^k k!\,(n-2k)!} \\ &= n! \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \frac{(\alpha x^2 + \beta)^{k-l} x^{n-3k+3l}}{(-C)^k k!(n-2k)!} \end{aligned}$$

$$
\begin{aligned}
&= n!\sum_{k=0}^{[\frac{n}{2}]}\sum_{l=0}^{k}\frac{(\alpha x^2+\beta)^{k-l}x^{n-3(k-l)}}{(-C)^k l!\,(k-l)!\,(n-2k)!} \\
&= n!\sum_{j=0}^{[\frac{n}{2}]}\sum_{l=0}^{[\frac{n}{2}]-j}\frac{(\alpha x^2+\beta)^{j}x^{n-3j}}{(-C)^{l+j} l!\,j!\,\{n-2(l+j)\}!} \\
&= n!\sum_{j=0}^{[\frac{n}{2}]}\sum_{l=0}^{[\frac{n-2j}{2}]}\frac{(\alpha x^2+\beta)^{j}x^{n-3j}}{(-C)^{l+j} l!\,j!\,\{(n-2j)-2l\}!} \\
&= n!\sum_{j=0}^{[\frac{n}{2}]}\frac{v_{n-2j}\left(1,\frac{1}{-C}\right)}{(n-2j)!}\frac{(\alpha x^2+\beta)^{j}x^{n-3j}}{(-C)^j j!}.
\end{aligned}
$$

Here if $C < 0$, then $v_{n-2j}(1, 1/(-C)) > 0$. For $f(x,t) = x^2$, a solution $u(x,t) = \sum_{n=0}^{N} c_n v_n(x,t)$ $(c_N \neq 0)$ satisfies

$$
x^2 = \sum_{k=0}^{N} c_n v_n\left(x, \frac{x^3+\alpha x^2+\beta}{-Cx}\right).
$$

By multiplying $x^{3[N/2]-N}$ to both sides and by comparing the coefficients, we see that $N = 2$. If $N = 2$, it is obvious that x^2 is not constructed by $v_n(x, (x^3+\alpha x^2+\beta)/(-Cx))$, $n = 0, 1, 2$.

If $C > 0$, then

$$
\begin{aligned}
v_n\left(x, \frac{x^3+\alpha x^2+\beta}{-Cx}\right) &= n!\sum_{j=0}^{[\frac{n}{2}]}\frac{v_{n-2j}\left(1,\frac{1}{-C}\right)}{(n-2j)!}\frac{(\alpha x^2+\beta)^{j}x^{n-3j}}{(-C)^j j!} \\
&= \frac{n!}{\sqrt{C}^n}\sum_{j=0}^{[\frac{n}{2}]}\frac{H_{n-2j}\left(\frac{\sqrt{C}}{2}\right)}{(n-2j)!j!}(-\alpha x^2-\beta)^j x^{n-3j}
\end{aligned}
$$

The highest degree term is

$$
\frac{H_n\left(\frac{\sqrt{C}}{2}\right)}{\sqrt{C}^n}x^n.
$$

If $\sqrt{C}/2$ is a zero point of the Hermite polynomial of degree m, then $f(x,t) = x^m$ is not constructed by $\{v_n(x, (x^3+\alpha x^2+\beta)/(-Cx))\}$.

If $\sqrt{C}/2$ is not a zero point of any Hermite polynomials, then we can follow the same argument in the case $C < 0$. Hence if ψ is of degree 3, then we see that the Basic Problem never holds. □

5 Equations of degree more than 3

In the case that $\deg\psi = N \geq 4$, we can show that the Basic Problem does not hold if $\psi(x,t) = Ax^N + t$, $\sum_{k=1}^{N} A_k x^k + Bxt$, $\sum_{k=1}^{N} A_k x^k t^{N-k}$, or $\sum_{k=0}^{[\frac{N}{2}]} A_k x^{N-2k} t^k$. We conjecture that the Basic Problem will not hold for any ψ of degree more than 3.

References

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（受理　平成14年３月19日）