愛知工業大学研究報告
第 42 号 A 平成 19 年

On Complex Analytic Mappings into Compact Riemann Surfaces

閉リーマン面への解析写像について

Yuji HASHIMOTO[†]

橋本有司

Abstract. We consider the complex analytic mappings of the Riemann surface $\hat{C}-E$ into the compact Riemann surface S of genus $g \geq 2$, where $\hat{C}$ is the extended complex plane and E is a totally disconnected compact set in the complex plane. We show that there exists no non-constant complex analytic mapping of $\hat{C}-E$ into S under some condition not depending on the logarithmic capacity of E.

1. Let E be a totally disconnected compact set in the complex z-plane C and let R be the complimentary domain $\hat{C}-E$ with respect to the extended complex plane $\hat{C}$. We consider the complex analytic mappings of R into S a compact Riemann surface of genus $g \geq 2$. According to Tsuji[10], if the logarithmic capacity of E is equal to 0, there exists no unramified complex analytic mapping of R into S. Further, according to Nishino[7] and Suzuki[8], if the logarithmic capacity of E is equal to 0, there exists no non-constant complex analytic mapping of R into S. In this paper, we shall show that, if E satisfies some appropiate condition, which is not depending on the logarithmic capacity of E, there exists no non-constant complex analytic mapping of R into S. The method used here is the one given by Carleson[1] and Matsumoto[5].

2. Let E, R and S be as in **1**. Let $\{R_n\}$ $(n = 0, 1, 2, \cdots)$ be an exhaustion of R with an additional condition such that each component $R_{n,k}$ $(k = 1, 2, \cdots, k_n)$ of $R_n - \overline{R}_{n-1}$ is doubly connected and branches off into at most ρ $(\rho \geq 1)$ components of $R_{n+1} - \overline{R}_n$. We denote by $\mu_{n,k}$ the harmonic modulus of $R_{n,k}$ and set $\mu_n = \min_{k=1,\cdots,k_n} \mu_{n,k}$. In these settings, we can state our theorem as follows.

Theorem. If $\lim_{n\to\infty} \mu_n = \infty$, then there exists no non-constant analytic mapping of R into S.

For the proof, the following lemma is essential.

Lemma. Let $f(z)$ be a complex analytic mapping of $G = \{1 < |z| < e^{\mu}\}$ into S. Then, the length L of the image $f(|z| = e^{\frac{\mu}{2}})$ with respect to the hyperbolic metric on S is dominated by $\dfrac{2\pi^2}{\mu}$.

Proof. Let $d\sigma_G$ and $d\sigma_S$ be the hyperbolic metrics on G and S induced by the Poincaré metric $\dfrac{2}{1-|\zeta|^2}|d\zeta|$ on the unit disk $|\zeta| < 1$ respectively. Then, we have

$$d\sigma_G = \frac{\pi}{\mu |z| \sin(\frac{\pi}{\mu}\log|z|)}|dz|.$$

According to the decreacing principle of the hyperbolic metric, we have $f^*d\sigma_S \leq d\sigma_G$, where $f^*d\sigma_S$ is the induced metric of $d\sigma_S$ by $f(z)$. Therefore, we have

$$L \leq \int_{|z|=e^{\frac{\mu}{2}}} \frac{\pi}{\mu|z|\sin(\frac{\pi}{\mu}\log|z|)}|dz| = \int_0^{2\pi} \frac{\pi}{\mu e^{\frac{\mu}{2}} \sin(\frac{\pi}{\mu}\log e^{\frac{\mu}{2}})} e^{\frac{\mu}{2}} d\theta = \frac{2\pi^2}{\mu}.$$

[†] Center for Genaral Education

Proof of the theorem. Let $\{R_n\}$ be an exhaustion of R. We may prove the theorem, without loss of generality, under the assumption that R_0 is simply connected and each component $R_{n,k}(k = 1,\cdots,2^n)$ branches off into two components $R_{n+1,2k-1}$ and $R_{n+1,2k}$. Now, let $f(z)$ be a complex analytic mapping of R into S. Accoding to the above lemma, as $R_{n,k}$ is conformally equivalent to the anulus $G = \{1 < |\zeta| < e^{\mu_{n,k}}\}$, there exists a simple closed curve $\Gamma_{n,k}$ in $R_{n,k}$ corresponding to the curve $|\zeta| = e^{\frac{\mu_{n,k}}{2}}$ such that the hyperbolic length $L_{n,k}$ of the image $f(\Gamma_{n,k})$ is dominated by $\dfrac{2\pi^2}{\mu_{n,k}}$.

We denote by $\Delta_{n,k}$ the triply connected domain bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ and consider the analytic mapping $f(z)$ in $\Delta_{n,k}$. By the condition of the theorem $\lim_{n\to\infty}\mu_n = \infty$ and the estimate of the lemma $L_{n,k} \le \dfrac{2\pi^2}{\mu_{n,k}} \le \dfrac{2\pi^2}{\mu_n}$, we can take an integer n_0 sufficiently large so that for $n \ge n_0$ the images $f(\Gamma_{n,k})$, $f(\Gamma_{n+1,2k-1})$ and $f(\Gamma_{n+1,2k})$ are contained in some sufficiently small schlicht hyperbolic disks $D_{n,k}$, $D_{n+1,2k-1}$ and $D_{n+1,2k}$ in S respectively. We call $f(z)$ nondegenerate in $\Delta_{n,k}$ if $f(z)$ takes the values outside of $D_{n,k} \cup D_{n+1,2k-1} \cup D_{n+1,2k}$ and we call $f(z)$ degenerate in $\Delta_{n,k}$ otherwise.

We shall show that the nondegenerate case cannot occur for $n \ge n_0$. We suppose that $f(z)$ is nondegenerate in $\Delta_{n,k}$ for some $n \ge n_0$. In the case where $D_{n,k}$, $D_{n+1,2k-1}$ and $D_{n+1,2k}$ are mutually disjoint, we can take the p-ply connected closed domain K_0 in $\Delta_{n,k}$ which is mapped properly onto the q-sheeted covering surface of $S - D_{n,k} \cup D_{n+1,2k-1} \cup D_{n+1,2k}$. According to the Hurwitz formula, we have $p-2 = q(2g+1)+v$, where $p-2$ and $2g+1$ are the Euler characteristics of K_0 and $S - D_{n,k} \cup D_{n+1,2k-1} \cup D_{n+1,2k}$ respectively and v is the sum of orders of the multiple points in K_0. Therefore, taking $g \ge 2$ into account, we have $p \ge 5q+2$. On the other hand, the boundaries of K_0 are mapped on the boundaries of $S - D_{n,k} \cup D_{n+1,2k-1} \cup D_{n+1,2k}$, so that we have $p \le 3q$, which is a contradiction. In the case where one of $D_{n,k}$, $D_{n+1,2k-1}$ and $D_{n+1,2k}$, say $D_{n,k}$, and the union of the other two $D_{n+1,2k-1} \cup D_{n+1,2k}$ are disjoint, we take a hyperbolic disk D_0 containing $D_{n+1,2k-1} \cup D_{n+1,2k}$ and apply the same argument. Taking the p-ply connected closed domain K_0 in $\Delta_{n,k}$ which is mapped properly onto the q-sheeted covering surface of $S - D_{n,k} \cup D_0$, we have $p-2 = q(2g)+v$, where $p-2$ and $2g$ are the Euler characteristics of K_0 and $S - D_{n,k} \cup D_0$ respectively and v is the sum of orders of the multiple points in K_0. Therefore, we have $p \ge 4q+2$. On the other hand, we have $p \le 2q$, which is a contradiction. In the case where $D_{n,k}$, $D_{n+1,2k-1}$ and $D_{n+1,2k}$ are not disjoint, we take a hyperbolic disk D_0 containing $D_{n,k} \cup D_{n+1,2k-1} \cup D_{n+1,2k}$ and apply the same argument. Taking the p-ply connected closed domain K_0 in $\Delta_{n,k}$ which is mapped properly onto the q-sheeted covering surface of $S - D_0$, we have $p-2 = q(2g-1)+v$, where $p-2$ and $2g-1$ are the Euler characteristics of K_0 and $S - D_0$ respectively and v is the sum of orders of the multiple points in K_0. Therefore, we have $p \ge 3q+2$. On the other hand, we have $p \le q$, which is a contradiction.

The above argument shows that $f(z)$ is degenerate in $\Delta_{n,k}$ for all $n \ge n_0$. We take $\Delta_{n_0,k}$ and connect $\Delta_{n_0+1,2k-1}$ and $\Delta_{n_0+1,2k}$ with $\Delta_{n_0,k}$ in the universal covering surface of S. Further, we connect $\Delta_{n_0+2,4k-3}$ and $\Delta_{n_0+2,4k-2}$ with $\Delta_{n_0+1,2k-1}$ and connect $\Delta_{n_0+2,4k-1}$ and $\Delta_{n_0+2,4k}$ with $\Delta_{n_0+1,2k}$ in the universal covering surface of S. Continuing this process successively, we can see that $f(z)$ is a complex analytic mapping of the end of R bounded by $\Gamma_{n_0,k}$ into the universal covering surface of S. Mapping the universal covering surface conformally onto the unit disk, we obtain a bounded analytic function in the end of R bounded by $\Gamma_{n_0,k}$. According to the Pfluger-Mori criterion, the subset $E_{n_0,k}$ of E contained in $\Gamma_{n_0,k}$ is the set of removable singularities for $f(z)$ $(k = 1,\cdots,2^{n_0})$. Therefore, $f(z)$ is a complex analytic mapping of $\hat{C}$ into S and becomes a constant.

3. We shall give some examples for which the above theorem is applicable and also consider the relation among the existence of non-constant complex analytic mappings of R into S, the existence of transcendental meromorphic functions on R with three Picard exceptional values and the existence of transcendental meromorphic functions on R with five totally ramified values.

Example 1. Let E be a Cantor set with successive ratios $\{\xi_n\}$. If $\lim_{n\to\infty}\xi_n = 0$, then the condition of the theorem is satisfied for $\hat{C} - E$, so that there exists no non-constant complex analytic mapping of $\hat{C} - E$ into S. As the condition of the logarithmic capacity of E being equal to 0 is $\sum_{n=1}^{\infty}\dfrac{\log\xi_n^{-1}}{2^n} = \infty$, we can give a Cantor set E of positive logarithmic capacity for which there exists no non-constant complex analytic mapping of $\hat{C} - E$ into S. Further, according to the results of

Matsumoto[6] and Toppila[9], taking a Cantor set E satisfying $\lim_{n\to\infty}\frac{\xi_{n+1}}{\xi_n}=0$, we can give the Cantor set E for which there exists no transcenental meromorphic function on $\hat{C}-E$ with three Picard exceptional values and no non-constant complex analytic mapping of $\hat{C}-E$ into S.

Example 2. (cf. Matsumoto[4]) Let $l_0>l_1>l_2>\cdots$ ($l_0<\frac{\sqrt{3}}{2}$, $l_{n+1}<\frac{l_n}{3}$) be a sequence of positive numbers satisfying $\lim_{n\to\infty}\frac{l_{n+1}}{l_n}=0$. We denote by $A(r_1,r_2,r_3)$ the surface $\hat{C}-\bigcup_{k=0}^{2}\{|z-e^{\frac{2k\pi}{3}i}|<r_{k+1}\}$ and by $B_k(r_1,r_2)$ the surface $\{r_2\le|z-e^{\frac{2k\pi}{3}i}|\le r_1\}$ with a slit joining $(1+\frac{2}{3}r_1-r_2)e^{\frac{2k\pi}{3}i}$ and $(1+\frac{2}{3}r_1+r_2)e^{\frac{2k\pi}{3}i}$ $(k=0,1,2)$. Let F_0 be the surface $A(l_0,l_0,l_0)$. We connect $B_k(l_0,l_1)$ $(k=0,1,2)$ with F_0 and denote the resulting 6-ply connected surface with three slits by F_1. Further, connecting $B_k(l_1,l_2)$ $(k=0,1,2)$ with F_1, we connect $B_0(l_0,l_1)\cup B_0(l_1,l_2)\cup A(l_0,l_1,l_1)\cup B_1(l_1,l_2)\cup B_2(l_1,l_2)$, $B_1(l_0,l_1)\cup B_1(l_1,l_2)\cup A(l_1,l_0,l_1)\cup B_0(l_1,l_2)\cup B_2(l_1,l_2)$ and $B_2(l_0,l_1)\cup B_2(l_1,l_2)\cup A(l_1,l_1,l_0)\cup B_0(l_1,l_2)\cup B_1(l_1,l_2)$ with $F_1\cup B_0(l_1,l_2)\cup B_1(l_1,l_2)\cup B_2(l_1,l_2)$ crosswise across the three slits joining $(1+\frac{2}{3}l_0-l_1)e^{\frac{2k\pi}{3}i}$ and $(1+\frac{2}{3}l_0+l_1)e^{\frac{2k\pi}{3}i}$ $(k=0,1,2)$. We denote the resulting 24-ply connected 4-sheeted covering surface of $A(l_2,l_2,l_2)$ with 12 slits by F_2. Continuing this process successively, we obtain the $6\cdot 4^{n-1}$-ply connected 4^{n-1}-sheeted covering surface F_n of $A(l_n,l_n,l_n)$ with $3\cdot 4^{n-1}$ slits and we denote the limit surface of F_n by F. Here, as the surface F is of planar character, by taking a suitable totally disconnected compact set E, we can map the surface F conformally onto $\hat{C}-E$. By the construction of the surface F, there exists a transcendental meromorphic function on $\hat{C}-E$ with three Picard exceptional values and as the condition of the theorem is also satisfied for $\hat{C}-E$, there exists no non-constant complex analytic mapping of $\hat{C}-E$ into S.

Example 3. (cf. Hashimoto-Matsumoto[2]) Let $l_0>l_1>l_2>\cdots$ be a sequence of positive numbers satisfying $\lim_{n\to\infty}\frac{l_{n+1}}{l_n}=0$. We denote by $C(r_1,r_2,r_3,r_4,r_5)$ the surface $\hat{C}$ with five slits joining $e^{\frac{2k\pi}{5}i}$ and $(1+r_{k+1})e^{\frac{2k\pi}{5}i}$ $(k=0,\cdots,4)$. Let F_0 be the surface $C(l_0,l_0,l_0,l_0,l_0)$. We connect $C(l_0,l_1,l_1,l_1,l_1)$, $C(l_1,l_0,l_1,l_1,l_1)$, $C(l_1,l_1,l_0,l_1,l_1)$, $C(l_1,l_1,l_1,l_0,l_1)$ and $C(l_1,l_1,l_1,l_1,l_0)$ with F_0 crosswise across the five slits joining $e^{\frac{2k\pi}{5}i}$ and $(1+l_0)e^{\frac{2k\pi}{5}i}$ $(k=0,\cdots,4)$ and denote the resulting 20-ply connected 6-sheeted covering surface of $\hat{C}$ with 20 slits by F_1. Continuing this process successively, we obtain the $5\cdot 4^n$-ply connected $(\frac{5}{3}(4^n-1)+1)$-sheeted covering surface F_n of $\hat{C}$ with $5\cdot 4^n$ slits and we denote the limit surface of F_n by F. As the surface F is of planar character, taking a suitable totally disconnected compact set E, we can map the surface F conformally onto $\hat{C}-E$. By the construction of the surface F, there exists a transcendental meromorphic function on $\hat{C}-E$ with five totally ramified values and as the condition of the theorem is also satisfied for $\hat{C}-E$, there exists no non-constant complex analytic mapping of $\hat{C}-E$ into S.

It is not known whether there exists a totally disconnected compact set E, for which there exists a non-constant analytic mapping of $\hat{C}-E$ into S and for which there exists no transcendental meromorphic function on $\hat{C}-E$ with three Picard exceptional values or with five totally ramified values. In this respect, we remark that there exists a Riemann surface R of infinite genus and with one ideal boundary, for which there exists a non-constant analytic mapping of R into S and for which there exists no non-constant meromorphic function on R with three Picard exceptional values (cf. Ozawa[3]).

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(受理　平成 19 年 3 月 19 日)