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Fast Algorithm for Short Time DFT Hilbert Transformers and Its Characteristics

Short Time DFTヒルベルト変換器の高速化アルゴリズムとその特性

岸　政七†,　　服部徳宏†
Masahichi KISHI,　Norihiro HATTORI

ABSTRACT *An ideal Hilbert transformer is realized as previously reported as the short time DFT Hilbert transformer by shifting the phase based on the concept of instantaneous spectra. Unfortunately, this ST DFT Hilbert transformer requires huge amount of computing power in analyzing instantaneous spectra, which becomes a severe problem on the developing stages. However, introducing the FFT structure makes possible to solve this problem to reduce the processing amount to allow to execute these processing on a single commercial DSP chip without degradation in its characteristics.*

1. INTORODUCTION

It is evident that a reduction in the frequency occupancy over each speech channel would result in more efficient utilization of radio frequency resources. Although it is the most efficient system for narrowing the frequency bandwidth occupancy, SSB has been suffered from such speech quality degradation as fading on multipass propagation.

However, once SSB modulated signals are combined with the carrier at an apriori level, new modulated RZ SSB signals are introduced as an angular modulating signals, which carry the modulated signals both in amplitude and phase. Therefore, RZ SSB systems have successfully improved the speech quality while reducing fading noise.

As well known, Hilbert transformers play a fundamental role in SSB or RZ SSB in eliminating one side band from the double side band of AM modulated signals without any distortions. Especially, the short time DFT (ST DFT) Hilbert transformers are based on instantaneous spectrum analysis [1]. Where instantaneous spectra are provided, the Hilbert transform is realized by merely exchanging the complex real or imaginary parts of these spectra with each other. The Hilbert transformed output signals are synthesized via an inverse ST DFT (ST IFT) of the interchanged instantaneous spectra.

2. STRUCTURAL CONSIDERATIONS

2.1 Principle of the ST DFT Hilbert Transformer

The ST DFT Hilbert transformer is featured of shifting input signal phase on the phase plane. All the input signals are at first analyzed by ST DFT to yield the instantaneous spectrum $\Phi(n)$ [2],

† 愛知工業大学　情報通信工学科（豊田市）

$$\Phi(n) = \{\ \phi_0(n)\ \phi_1(n)\ \phi_2(n) \cdots \phi_{N-1}(n)\ \}^T. \quad (1)$$

Where $\phi_k(n)$ is the spectral component at the frequency index $k$of $\Phi(n)$ given as

$$\phi_k(n) = \sum_{r=-\infty}^{\infty} h(n-r)\,x(r)\,W_N^{-rk}. \quad (2)$$

Here, $k$is $0 \le k < N$; $x(r)$ is the input data at the sampling time r; W_N^{-rk} is the same operators as that defined in the existing DFT, $W_N^{-rk} = exp\left(-j\frac{2\pi rk}{N}\right)$; $h(*)$ is the significant window functions gives as an example which truncated by 2m frame number,

$$h(p) = \frac{\sin(p\pi/N)}{p\pi/N}, \quad -mN \le p \le mN. \quad (3)$$

More detailed discussions are shown by the authors in ref.3.

Where instantaneous spectra are provided, the Hilbert transform is realized by merely exchanging the complex real or imaginary parts of these spectra with each other. This Hilbert transformed spectrum $\widehat{\Phi}(n)$are necessary to hold causality with restricting complex conjugate relation between each $\widehat{\phi_k}(n)$and $\widehat{\phi}_{N-k}(n)$as being mirror symmetric with pivot at index $N/2$. The output signal $\widehat{y}(n)$is finally synthesized through an ST IFT from the transformed spectrum $\widehat{\Phi}(n)$

$$\widehat{y}(n) = \frac{2}{N}\sum_{k=1}^{N/2-1} Real\left\{\widehat{\phi_k}(n)\,W_N^{nk}\right\}. \quad (4)$$

2.2 Unit Sample Response

The unit sample response $I_s(n)$ of an ST DFT Hilbert transformer is given by eq.5 when the unit sample $x(n)$ is taken to be $x(0)=1$.

$$I_s(n) = \begin{cases} \frac{2}{N}\cdot\frac{\sin(2\pi n/N)}{1-\cos(2\pi n/N)}\cdot\frac{\sin(\pi n/N)}{\pi n/N} \\ \quad = \frac{2}{\pi n}\cos(\pi n/N), \ for\ odd\ n \\ 0, \quad for\ even\ n. \end{cases} \quad (5)$$

Equation 5 shows the ideal unit sample response of a Hilbert transformer if the Nyquist window frame number is infinity, because the frequency domain Hilbert transform operation excludes all the processing errors and a Nyquist window of infinite length acts as an ideal low-pass filter as shown by the Nyquist window factor $\sin(\pi n/N)/(\pi n/N)$.

Equation 6 provides the unit sample response $I_m(n)$ of Rabiner's minmax FIR (minmax) Hilbert transformer [4].

$$I_m(n) = \frac{2\sin^2(\pi n/2)}{\pi n} = \begin{cases} \frac{2}{\pi n}, & for\ odd\ n \\ 0, & for\ even\ n. \end{cases} \quad (6)$$

It is clear from eqs.5 and 6 that a ST DFT Hilbert transformer enhances the performance of a minmax Hilbert transformer. That is,

$$\lim_{N\to\infty} I_s(n) = \lim_{N\to\infty}\frac{2\cos(\pi n/N)}{\pi n} = \frac{2}{\pi n} = I_m(n), \quad for\ odd\ n. \quad (7)$$

3 FAST PROCESSING FOR ST DFT HILBERT TRANSFORMERS

3.1 Fast Algorothm Based on the FFT Structure

ST DFT Hilbert transformers are free from any distortions owing to exchanging complex real or imaginary parts of instantaneous spectra $\Phi(*)$ with each other. However, they are

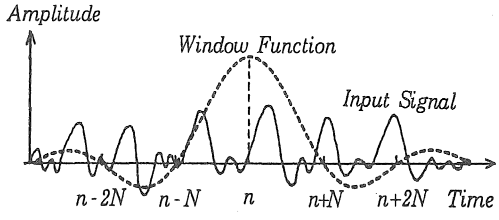

Fig.1 Window function superimposed on input signal.

suffered from great amount of computing power in analyzing input signals via a ST DFT.

Equation 2 is modified by changing variable $r = lN + m$ as follows.

$$\phi_k(n) = \sum_{l=-\infty}^{\infty}\sum_{m=0}^{N-1} h(n - lN - m)x(lN + m)W_N^{-(lN+m)k} = \sum_{m=0}^{N-1} \widetilde{x_m}(n)\, W_N^{-mk}. \quad (8)$$

here,

$$\widetilde{x_m}(n) = \sum_{l=-\infty}^{\infty} h(n - lN - m)x(lN + m) \quad (9)$$

$$W_N = e^{j\frac{2\pi}{N}}.$$

The instantaneous spectrum $\Phi(n)$ is defined by

$$\Phi(n) = W_N\widetilde{X}(n) \quad (8)'$$

where matrix W_N is time invariant defined as,

$$W_N = \begin{bmatrix} W_N^{-0} & W_N^{-0} & W_N^{-0} & \cdots & W_N^{-0} \\ W_N^{-0} & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ W_N^{-0} & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^{-0} & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} \quad (10)$$

and $\widetilde{X}(n)$ is time variant given by

$$\widetilde{X}(n) = \begin{bmatrix} \sum_{l=-\infty}^{\infty} x(lN)\,h(n - lN) \\ \sum_{l=-\infty}^{\infty} x(lN+1)\,h(n - lN - 1) \\ \sum_{l=-\infty}^{\infty} x(lN+2)\,h(n - lN - 2) \\ \vdots \\ \sum_{l=-\infty}^{\infty} x(lN+N-1)\,h(n - lN - N + 1) \end{bmatrix} \quad (11)$$

or

$$\widetilde{X}(n) = \{\, \widetilde{x_0}(n)\, \widetilde{x_1}(n)\, \widetilde{x_2}(n) \cdots \widetilde{x_{N-1}}(n) \,\}^T, \quad (11)'$$

Attentions must be paid on analyzing input signals via DFT given by eqs.8 or 8', because there exists two time axes, the one is absolute time n to define time base for input signal and the other is temporary time m which is free from absolute time base in setting observation system on arbitrary absolute time base. That is, the origin of time base in observation system is defined by a time when the parameter of window function $h(*)$ becomes to zero at every absolute time $n = 0 \bmod N$ to be shown in m_{th} row in eq.11. Among these operations given by eq.8', the DFT operators are static on the observation time base.

Where the input vector $\widetilde{X}(n)$ is set on the absolute time base, equation 8' is modified by introducing FFT structure as follows.

$$\Phi(n) = W_N R^n X(n) \quad (12)$$

$$X(n) = \{\, x_0(n)\, x_1(n)\, x_2(n) \cdots x_{N-1}(n) \,\}^T, \quad (13)$$

and $x_m(n)$ is given by

$$x_m(n) = \sum_{l=-\infty}^{\infty} h(-lN - m)\,x(lN + m). \quad (14)$$

R is $N \times N$ square matrix of left row shifting operator as

$$R = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & & & & & & & \\ & & & \cdot & & & & & & \\ & & & & \cdot & & & & & \\ & & & & & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{bmatrix} \quad (15)$$

$X(n)$ is illustrated in fig.1 to be calculated on the absolute time by settling the observation origin on arbitrary time n. As shown in eq.13 for an example of input vector $X(n)$, the origin point of observation system stands on the first row of $X(n)$. Therefore, an instantaneous spectrum $\Phi(n)$ is shown to be calculated in similar to the existing DFT only with settling the origin point of $h(*)$ to the desired absolute time n. Adjustment between the absolute and observation time bases is executed only with operator R^n.

It is easy to understand without any discussions except this row rotation R^n that eq.12 is directly performed with DFT formulation in order to allow introduction of FFT structure.

3.2 Fast Algorithm Based on the Frequency Domain Interpolation

Where a little degradation is permitted in fast processing from practical application of view, $\widehat{\phi_k}(n)$ may be reproduced from decimated spectrum component $\widehat{\phi_k}(rR)$ at every R sampling clocks as follows,

$$\widehat{\phi_k}(n) = \sum_{r=L^-}^{L^+} f(n-rR)\widehat{\phi_k}(rR) \qquad (16)$$

where $f(n-rR)$ is, for example, Lagrange interpolation function of $2Q$ frame given as

$$f(n-rR) = \frac{-1^{r+Q}}{(Q-1+r)!\,(Q-r)!\,(\frac{n}{R}-r)} \prod_{i=1}^{Q}\left(\frac{n}{R}+Q-i\right) \qquad (17)$$

$$here,\quad L^+ = \left[\frac{n}{R}+Q\right],\quad L^- = \left[\frac{n}{R}\right]-Q+1\,, \qquad (18)$$

$[*]$ represents the largest integer contained $*$.

The Hilbert transformed signal $\widehat{y}(n)$ is consequently synthesized via ST FFT from the interpolated spectrum $\widehat{\Psi}(n)$ as follows.

$$\widehat{y}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{\phi_k}(n)\, W_N^{nk} \qquad (19)$$

Equation 19 gives a fast algorithm as shown in fig.1 based on the frequency domain interpolation. The processing power is reduced to the almost same order of the FFTly, $N\log N$.

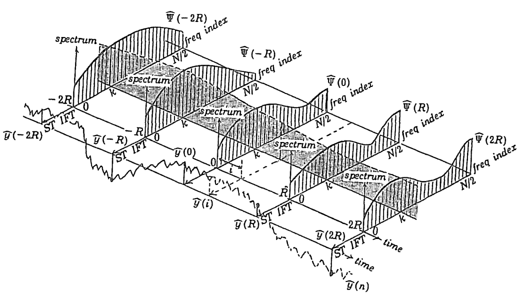

Fig.2 Fast processing diagram in the ST DFT Hilbert transformers based on frequency domain interpolation.

3.3 Fast Algorithm Based on the time Domain Interpolation

Substituting eq.16 into ST DFT, the output signal $\widehat{y}(n)$ is given as

$$\widehat{y}(n) = \sum_{k=0}^{N-1}\sum_{r=L^-}^{L^+} f(n-rR)\widehat{\phi_k}(rR)\, W_N^{nk} \qquad (20)$$

Since all the operations in the above equation are linear on the finite operand for k and r, eq. 20 stands on exchanging the summation order. That is,

$$\widehat{y}(n) = \sum_{r=L^-}^{L^+} f(n-rR)\,\widehat{s_r}(n) \qquad (21)$$

Where

$$\widehat{s_r}(n) = \sum_{k=0}^{N-1} \widehat{\phi_k}(rR)\, W_N^{nk}, \qquad L^- \le r \le L^+. \qquad (22)$$

Equation 21 shows that $\widehat{y}(n)$ is reproduced via time domain interpolation to yield faster algorithm than that of based on the frequency domain interpolation.

4.EXPERIMENTAL RESULTS

The validity of adopting fast algorithms into ST DFT Hilbert transformer was proven through computer simulation. Owing to employing interpolation, the system function of the fast ST DFT Hilbert transformer becomes to vary both with input timing of the unit sam-

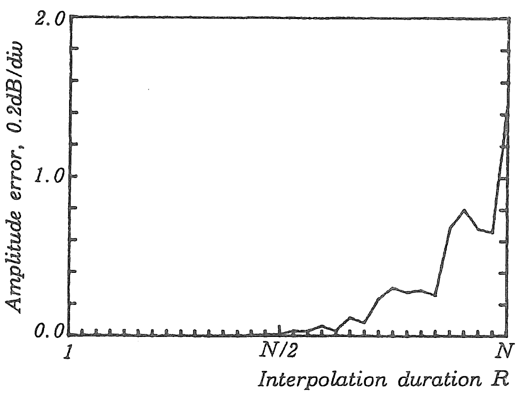

Fig.3 Mean absolute amplitude error of the ST DFT Hilbert transformers.

ple and with interpolation duration R . So long as the unit sample is given at the origin sampling clock, i.e., $x(n) = \delta(n-\tau)$, here $n = 0$, $\tau = 0$,the precision of transformers is constant in regardless Rvalue. While, if τ goes to non-zero, frequency responses are remarkable degraded as shown in figs.3 and 4 in taking R as a parameter when R being set up to the maximum value$N$. Amplitude error keeps peak values within the envelope of the minimax Hilbert transformer a shown in fig.3, where vertical axis is indicated by mean absolute of the amplitude error for all $0 \leq \tau < R$. Phase shifting error is scared beyond 0.05 degree in mean absolute error as shown in fig.4.

While interpolation duration R is set to be below $N/2$, the amplitude error is improved to be below 0.01 dB and phase shifting error is improved to be below 0.01 degree.

5.CONCLUSION

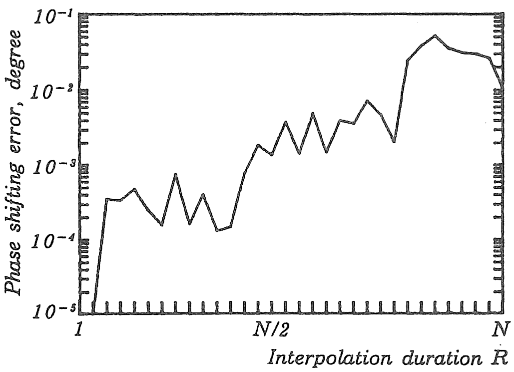

Fig.4 Mean absolute phase shifting error of the ST DFT Hilbert transformers.

A noble Hilbert transformer was discussed with emphasis on the concept of the instantaneous spectrum through fast algorithm and frequency responses. It was also shown through computer simulations that ST DFT Hilbert transformer is so precise as being equal to the existing Rabiner's pre-optimaized minimax both in phase and amplitude response.

Multi-rate sampling have been successfully introduced into the ST DFT Hilbert transformer in order to reduce computing power without almost any distortion, where the interpolation duration R is below $N/2$.

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